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# Quantum Effects on Cosmic Scales as an Alternative to Dark Matter and Dark Energy

**Da-Ming Chen [1,2,*] and Lin Wang [3]**

1 National Astronomical Observatories, Chinese Academy of Sciences, 20A Datun Road, Chaoyang District, Beijing 100101, China
2 School of Astronomy and Space Science, University of Chinese Academy of Sciences, Beijing 100049, China
3 School of Physics and Electronics, Henan University, Kaifeng 475004, China; wl010@bao.ac.cn
* Correspondence: cdm@nao.cas.cn

**Abstract:** The spin-torsion theory is a gauge theory approach to gravity that expands upon Einstein's general relativity (GR) by incorporating the spin of microparticles. In this study, we further develop the spin-torsion theory to examine spherically symmetric and static gravitational systems that involve free-falling macroscopic particles. We posit that the quantum spin of macroscopic matter becomes noteworthy at cosmic scales. We further assume that the Dirac spinor and Dirac equation adequately capture all essential physical characteristics of the particles and their associated processes. A crucial aspect of our approach involves substituting the constant mass in the Dirac equation with a scale function, allowing us to establish a connection between quantum effects and the scale of gravitational systems. This mechanism ensures that the quantum effect of macroscopic matter is scale-dependent and diminishes locally, a phenomenon not observed in microparticles. For any given matter density distribution, our theory predicts an additional quantum term, the quantum potential energy (QPE), within the mass expression. The QPE induces time dilation and distance contraction, and thus mimics a gravitational well. When applied to cosmology, our theory yields a static cosmological model. The QPE serves as a counterpart to the cosmological constant introduced by Einstein to balance gravity in his static cosmological model. The QPE also offers a plausible explanation for the origin of Hubble redshift (traditionally attributed to the universe's expansion). The predicted luminosity distance–redshift relation aligns remarkably well with SNe Ia data from the cosmological sample of SNe Ia. In the context of galaxies, the QPE functions as the equivalent of dark matter. The predicted circular velocities align well with rotation curve data from the SPARC (Spitzer Photometry and Accurate Rotation Curves database) sample. Importantly, our conclusions in this paper are reached through a conventional approach, with the sole assumption of the quantum effects of macroscopic matter at large scales, without the need for additional modifications or assumptions.

**Keywords:** alternative theories of gravity; dark matter; dark energy; galactic rotation curves

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## 1. Introduction

The primary challenge facing the current theory of gravity, the Einstein–Newton theory, lies in the unresolved mysteries of dark matter and dark energy. To date, there has been a lack of direct observational evidence confirming the existence of these enigmatic components. Extensive efforts are being made to address this topic. On the theoretical front, much attention is directed towards modifying or extending the Einstein–Newton theory of gravity to align with astronomical observations. For instance, one can enhance the standard Lagrangian in general relativity by incorporating higher-order curvature corrections [1–6], or formulate non-linear Lagrangians [7,8]. Other relevant examples include modified Newtonian dynamics (MOND) [9,10] and its relativistic version [11], as well as conformal gravity [12,13].

In this paper, we explore the concept of quantum spacetime at cosmic scales as a potential alternative to dark matter and dark energy. This research represents a significant

extension of our previous study [14]. Rather than relying on previous assumptions, we explicitly propose scale-dependent quantum properties of spacetime. This proposition is motivated by our interest in potentially moving away from the notion of a preferred absolute spacetime (PAS) for the universe, as implied by general relativity (GR) and illustrated in the standard ΛCDM cosmology. It is believed that the PAS is theoretically characterized by the Friedmann–Lemaître–Robertson–Walker (FLRW) metric and empirically supported by the cosmic microwave background (CMB). One aim of this study is to demonstrate that the notion of a PAS is not inherently valid when considering the quantum effects of macroscopic matter at large scales.

Before going further, it is imperative to clarify our actual motivation in order to prevent any possible misunderstandings. For ease of reference, we categorize the universe into three distinct classes based on spatial scales: the cosmic scale (encompassing galaxies to the entire universe), the macroscopic scale (ranging from everyday life to the size of the solar system), and the microscopic scale (where quantum mechanics becomes significant, as is well-known). According to the mainstream view, the quantum effect for objects at the macroscopic and cosmic scales can be safely disregarded, due to their large mass. This is known as the classical limit. We will show, however, that quantum effects are dependent on the scale of the gravitational system considered and that the "classical limit" is only reached at macroscopic scale. In other words, quantum effects are significant not only at microscopic scale but also at cosmic scale, and can only be ignored at macroscopic scale. Intuitively, when we study cosmology, any observers who perceive distant galaxies in the universe, would expect them to exhibit quantum behavior resembling that of microscopic particles, due to their distant location. As for galactic dynamics, quantum effects should also be taken into account for galaxies, although in the central part, they can be ignored due to the mechanism presented in this study.

It is helpful to discuss the reasons behind generalizing the quantum effects observed in microscopic particles to macroscopic ones, as well as why the strength of these generalized quantum effects appears to increase with the spatial scale, based on our previous paper [14]. It is believed that gravity turns disorder into order, and order is fundamental to space, time, and spacetime—or, as we might say, spacetime is nothing but the continuous ordering of events. Thus, without gravity, there would be no spacetime. In any local inertial frame, gravity is still present, but the net gravitational force is canceled out by the inertial force, leaving order or flat spacetime behind. In special relativity, rigid rods are used to create coordinate lattices, ensuring that every event has a position at any given time. According to GR, the global structure of curved spacetime can still be described using arbitrarily curved rigid rods (depending on the matter distribution) of arbitrary length. However, quantum mechanics reveals that microscopic particles can escape the order depicted by spacetime and thus elude the control of gravity. The quantum effects of electrons, for instance, can be attributed to their quantum randomness (also known as intrinsic randomness), which is markedly different from classical randomness (or apparent randomness), exemplified by Brownian motion. A free electron deviates from a straight line or a curved geodesic, which implies that gravity is not strong enough to bring it into order. This explains quantum effects from a geometric point of view. Gravity does bring macroscopic matter into order in the sense that any free macroscopic object will move along a straight line or a curved geodesic. This fact is fundamental to both Newtonian theory and GR. However, if we assume that all matter, irrespective of its mass, has two opposite properties—gravity and quantum randomness—then it is possible that gravity is not strong enough to bring any distant macroscopic object into order, just as occurs with microscopic particles. If our assumption is true, then general relativity (GR) is valid only approximately within a sufficiently small neighborhood of any point in the spacetime manifold. In this case, the uncertainty in matter distribution, originating from quantum randomness, will accumulate with spatial distance. Or, put another way, the strength of quantum effects appears to increase with spatial scale. On the other hand, since the spin-induced torsion arising from macroscopic matter vanishes locally, the geodesics of test particles are precisely defined.

This serves as the physical foundation for physicists to construct coordinate lattices. As a result, in general, spacetime is not only curved but also flexible. It turns out that the preferred absolute spacetime (PAS), which is determined by all matter in the universe, loses its meaning when scale-dependent quantum effects come into play. Any new fundamental assumptions about physical laws should be falsifiable. The rest of this study represents a first attempt along this line.

We will illustrate the integration of our assumption with the Einstein equations. It is natural for us to generalize the Dirac theory to describe the macroscopic matter content of gravitational systems with cosmic scale.

There are two approaches to interpreting quantum mechanics. The first is the standard view, which asserts that microscopic particles cannot have continuous trajectories and that non-commuting observables (such as position and momentum) must satisfy the uncertainty principle. The second approach is known as the causal interpretation, which posits that each microscopic particle has a continuous trajectory and replaces the uncertainty principle with the concept of quantum potential [15,16]. Both approaches yield equivalent predictions for experimental results. However, the causal interpretation, particularly in its geometric algebra (GA) version, provides profound insights into the quantum nature of spacetime and matter [17–20]. In this paper, similarly to in our previous work, we utilize spacetime algebra (STA) [20–22] as our mathematical language. STA is constructed from a Minkowskian vector space and provides a straightforward geometric understanding of Dirac theory. It provides a natural link to classical mechanics, and for expressing Dirac theory with observables, it offers enhanced computational efficiency and capabilities [18] when compared to the tensor analysis method [23]. The fundamentals of STA are outlined in Appendix A. In the STA version of Dirac theory, known as real Dirac theory, Hestenes identified the complex number present in the matrix version of Dirac theory as the spin plane [17]. Furthermore, since the complex number is always accompanied by the Planck constant $\hbar$, a rigorous derivation of the Schrodinger theory from the Pauli or Dirac theory implies that the Schrodinger equation describes an electron in an eigenstate of spin, rather than, as commonly believed, an electron without spin [24]. Correspondingly, there are two gauge theories of gravity concerning our research work: one is the tensor analysis approach [25], and the other is the gauge theory gravity (GTG) developed by the Cambridge group using STA [26]. These two theories are nearly equivalent, but the latter is conceptually clearer and technically more powerful for comprehending and calculating the challenges encountered in quantum mechanics and gravity. Therefore, we choose to adopt GTG in this paper. The fundamentals of GTG and its applications to spin-$\frac{1}{2}$ particles are summarized in Appendix C.

It is now widely acknowledged that the quantum random motion of spin-$\frac{1}{2}$ particles can be fully described by their spin. In the presence of gravity, the spin gives rise to the torsion of spacetime [21,25–29]. In particular, the effects of spin-torsion in GTG were investigated in Ref. [27]. GTG is nearly equivalent to Einstein–Cartan gravity [25]. The stress-energy tensor derived from the Dirac theory contains an asymmetric component, representing the contribution of quantum spin. As a consequence, the metric of spacetime as predicted by the generalized Einstein equations includes a component that accounts for the torsion of spacetime. As will be shown later, the torsion term appearing in the metric represents a modification to GR. To date, the problem of finding solutions for a Dirac field coupled to gravity in a self-consistent manner has primarily been considered for microscopic particles. To our knowledge, there is only one work that explored the massive, non-ghost cosmological solutions for the Dirac field coupled self-consistently to gravity [30], which is close to our present study. As expected, the authors applied their methods to the very early universe, with the Dirac field describing massive yet microscopic particles. This differs from our present study, in which we assume that the Dirac field describes macroscopic particles within a static universe.

For massive fermions, such as electrons and neutrons, the mass will always appear in the phase factor of the solutions of wave equations. When gravity matters, this mass dependence remains. Consequently, it is generally believed that the gravity effect is not

purely geometric at the quantum level. Hence, the immediate question is how can we reconcile, if possible, the contradiction between the quantum effect and the requirement of a geometric description of gravity? For microparticles, reconciling this contradiction is indeed impossible. However, in this study, we propose that the scale-dependent quantum effect on macroscopic matter only becomes significant at cosmic scales. Locally, at a macroscopic scale, the large mass $m$ of each particle, or equivalently, the small value of the number density $\rho$ of the fluid, guarantees what is known as the classical limit, allowing for a geometric description of the gravitational effect on macroscopic matter. The remaining question is what mechanism ensures that quantum effects are significant at cosmic scales while being negligible at macroscopic scales?

The answer to this question is not intricate, but rather subtle. The specifics will be provided later, but for now, we will outline the main concepts behind the mechanism. Our analysis commences with an examination of the Dirac theory within various inertial frames [18]. Generally, the Lorentz–invariant Dirac spinor is defined in spacetime as

$$\psi(x) = \rho^{1/2} e^{i\beta(x)/2} R(x). \tag{1}$$

where $\rho(x)$ is a scalar representing the proper probability density and $R(x)$ is a rotor (Lorentz rotation) satisfying $R\tilde{R} = 1$. In Dirac theory, the parameter $\beta(x)$ is intriguing. It is noteworthy that the states of a plane-wave particle have $\beta = 0$, while those of an antiparticle have $\beta = \pi$. In this paper, we aim to generalize the spinor $\psi(x)$ to describe free-falling macroscopic, non-relativistic particles with identical mass $m$. Therefore, it is logical to set $\beta = 0$ throughout. We thus adopt [14]

$$\psi(x) = \rho(x)^{1/2} R(x). \tag{2}$$

For our purpose, we can interpret $\rho = \psi\tilde{\psi}$ as the proper number density of a fluid, then we refer to $\rho_m = m\psi\tilde{\psi}$ as the corresponding proper mass density. As illustrated in Appendix B, the rotor $R$ can be used to transform a fixed frame $\{\gamma_\mu\}$ into a new frame $\{e_\mu = R\gamma_\mu\tilde{R}\}$, and we identify $v = e_0$ as the proper velocity associated with the expected history $x(\tau)$ of a particle and thus the current velocity of the fluid. Naturally, the corresponding stress–energy tensor of the fluid can be written as

$$T(a) = \rho_m a \cdot vv. \tag{3}$$

It is important to point out that this stress–energy tensor, which originates from the Dirac spinor given by Equation (2) for a single electron, definitely describes a classical pressure-free ideal fluid without incorporating quantum spin. Quantum spin is only included if the spinor satisfies the Dirac equation, as will be discussed soon. In this sense, we can say that the spinor given in Equation (2) captures all aspects of macroscopic particles, when considered with or without the Dirac equation.

The Dirac equation is

$$\hbar\nabla\psi i\gamma_3 = m\psi. \tag{4}$$

where $\hbar$ is the Planck constant. This equation describes a single spin-$\frac{1}{2}$ free particle with a fixed mass $m$ and a probability density $\rho$. It is convenient to define a spin density trivector as (note that this is denoted with $S_3$ in Appendix B)

$$S = \frac{\hbar}{2}\psi i\gamma_3\tilde{\psi} = \frac{\hbar}{2}\rho R i\gamma_3\tilde{R}. \tag{5}$$

We show in Appendix B that, from the Dirac Equation (4), the general stress–energy tensor for the spinor field is

$$T(a) = \rho_m a \cdot vv + [a \cdot \nabla(Sv)] \cdot v - (a \wedge \nabla) \cdot (Sv)v. \tag{6}$$

When $S = 0$ (this is the classical limit), this equation becomes the stress–energy tensor of a pressureless ideal fluid without spin, as shown in Equation (3). Clearly, if we treat each free-falling macroscopic particle with a fixed mass $m$ as a single spin-$\frac{1}{2}$ free particle, we have the opportunity to include its quantum behavior under certain circumstances. This is the main aim of this study.

We are now in a position to discuss how the spinor approach can unify quantum mechanics and classical mechanics. We believe this will be helpful to understand the physical significance of our proposal in this study. The general form of the Dirac spinor, as defined in Equation (1), and the corresponding Dirac equation are translations of the traditional matrix version of Dirac theory, expressed in terms of STA [18]. They are equivalent in practical applications. The advantages of the STA version of Dirac theory include providing an explicit geometric and causal interpretation of the theory, which makes it very convenient when incorporating with general relativity (GR). One of the unique features of the Dirac equation is that it allows for solutions with both positive and negative energy states. The negative energy solutions in the Dirac equation lead to the concept of the Dirac sea, a theoretical model that was used to explain the existence of antiparticles. Because of the presence of negative energy states, when dealing with Dirac spinors, one encounters different types of physical quantities called densities, such as scalar densities, which are Lorentz-invariant. Scalar densities can interact with scalar fields, and the Higgs field is one such example, which is responsible for giving mass to particles through their interaction with it. When we generalize the Dirac theory to describe classical macroscopic particles, however, we restrict ourselves to the positive energy states. This is adequate because, in the case of the non-relativistic energy of macroscopic particles, there are no particle–antiparticle creation processes involved. That is, we treat each macroscopic particle as a single Dirac fermion. As such, we recognize the proper number density $\rho = \psi\tilde{\psi}$ and the proper mass density $\rho_m = m\psi\tilde{\psi}$ (as measured by observers who are comoving with the current velocity $v = R\gamma_0\tilde{R}$) as that for the ensemble of a single macroscopic particle, described by the Dirac spinor give in Equation (2). As mentioned, classical mechanics are recovered when $S = 0$.

When applying these results to gravitational systems, we simply need to replace $\nabla$ with the covariant derive symbol $D$, while all other quantities remain gauge-invariant. As such, the classical limit (also known as the short wave approximation) refers to the fact that the contribution of the spin to the stress–energy is negligible. Clearly, the magnitude of the spin $|S| = \frac{\hbar}{2}\rho$ is determined by the number density $\rho$. Therefore, an alternative interpretation of the classical limit is that the value of the number density, $\rho$, is ignorably low, whereas a significant quantum effect arises when $\rho$ is sufficiently high. For a fixed mass density $\rho_m(x)$, the spin $|S|$ can be determined via $\rho(x) = \rho_m(x)/m$. This freedom strongly indicates that, in order to incorporate the scale-dependent quantum effect at scale, we can replace the mass $m$ in the Dirac equation with a mass function that depends on the spatial scale $\lambda$ of the system, denoted as $m(\lambda)$. The mass function $m(\lambda)$ naturally satisfies the condition that, when $\lambda \to 0$, then $S \to 0$, corresponding to the macroscopic scale. Conversely, when $\lambda \to \infty$, then $S$ becomes a $\lambda$-dependent quantity related to the constant density of the universe. The intermediate functional form of $m(\lambda)$, which plays a crucial role in the study of galaxies, can be determined through observations. Consequently, we can employ the Dirac equation in the presence of gravity without any modifications, as there are no derivatives of $m(\lambda)$ involved in our calculations. The properties of the mass function $m(\lambda)$ only need to be discussed in the final results.

It is important to explicitly acknowledge that when applying the Dirac equation to gravitational systems at cosmic scales, we shall show in Section 3 that the anisotropy of the spin density adheres to the symmetric properties of mass density $\rho_m(x)$. This fact only applies to macroscopic matter according to our assumption, where the gravitational effect is geometric throughout the spacetime manifold and, furthermore, the proper mass density and the spin density have already been averaged and must satisfy the condition that the quantum effect vanishes locally. In contrast, when considering electrons, the anisotropy of spin must be taken into account in all cases, as the gravitational effect is not purely

geometric. Thus, the quantum effect can never disappear locally, as is commonly understood [30–33].

The subsequent sections present an examination of radially symmetric and static gravitational systems based on our new postulation in Section 2, the applications of our findings to cosmology and galaxies in Section 3, and summarized conclusions and discussions in Section 4. We employ natural units ($G = \hbar = c = 1$) throughout, except where stated otherwise.

## 2. Radially Symmetric Gravitational Systems

We want to investigate the quantum nature of spacetime at cosmic scales, with the main subjects of concern being galactic dynamics and cosmology. Traditionally, a pressureless ideal fluid has been a good model for both galactic dynamics and matter-dominated era cosmology. In both scenarios, matter experiences free-fall in a gravitational field and is characterized by the density distribution $\rho(x)$ and velocity field $v(x)$, where $x$ represents the position vector in spacetime. This model is usually known as collapsing dust. In gravitational systems, it is widely recognized that a static state can only be sustained when there is a substantial gravitational potential well, as observed in galaxies. However, in the realm of cosmology, the homogeneous and isotropic distribution of matter across the entire universe renders static solutions non-existent.

Nevertheless, the introduction of quantum effects at cosmic scales fundamentally alters the situation.

The Dirac theory for radially symmetric gravitational systems can be investigated using gauge theory gravity (GTG). The latter is constructed such that the gravitational effects are described by a pair of gauge fields, $\bar{h}(a) = \bar{h}(a, x)$ and $\omega(a) = \omega(a, x)$, defined over a flat Minkowski background spacetime [26], where $x$ is the STA position vector and is usually suppressed for short. Luckily, the majority of the necessary results for our present study have already been derived in prior work [27]. These results are valid for microscopic particles; what we need to do is to generalize these to macroscopic matter.

Let us consider a radially symmetric and static gravitational system composed of free-falling particles with the identical mass $m$. We make the assumption that the Dirac spinor $\psi(r) = \rho(r)^{1/2} R(r)$ defined in (2) can fully capture all the essential physical aspects of the system if it satisfies the Dirac equation

$$D\psi i\gamma_3 = m\psi. \tag{7}$$

We interpret $\rho = \psi\tilde{\psi}$ as the proper number density and $\rho_m = m\psi\tilde{\psi}$ as the proper mass density. We adopt $S = \frac{1}{2}\psi i\gamma_3\tilde{\psi} = \frac{1}{2}\rho R i\gamma_3\tilde{R}$ defined in (5) as the spin density trivector for the gravitational system. So there are four variables, namely $m$, $\rho$, $\rho_m$, and $S$, that can be used to describe the gravitational system. However, only two of them are independent, as they must satisfy the conditions $\rho = \rho_m/m$ and $|S| = \frac{1}{2}\rho$. In this paper, we eliminate $m$ and retain the other three variables. Among them, the relationship between $\rho_m$ and $\rho$ can be further determined through observations by assuming a specific form for the mass function $m(\lambda)$. Once this has been carried out, we are left with only one variable, which is the proper mass density $\rho_m$. Therefore, if we are given the proper mass density $\rho_m$ of a gravitational system, we can predict the entire set of results, including the quantum effects.

Now we define a set of spherical coordinates. From the position vector of the flat spacetime

$$x = t\gamma_0 + r\sin\theta(\cos\phi\gamma_1 + \sin\phi\gamma_2) + r\cos\theta\gamma_3, \tag{8}$$

we obtain the basis vectors, as follows:

$$\begin{aligned}
e_t &= \partial_t x = \gamma_0, \\
e_r &= \partial_r x = \sin\theta(\cos\phi\gamma_1 + \sin\phi\gamma_2) + \cos\theta\gamma_3, \\
e_\theta &= \partial_\theta x = r\cos\theta(\cos\phi\gamma_1 + \sin\phi\gamma_2) - r\sin\theta\gamma_3, \\
e_\phi &= \partial_\phi x = r\sin\theta(-\sin\phi\gamma_1 + \cos\phi\gamma_2).
\end{aligned} \tag{9}$$

Since $e_\theta$ and $e_\phi$ are not unit, we define

$$\hat{\theta} \equiv e_\theta / r, \quad \hat{\phi} \equiv e_\phi / (r \sin\theta). \tag{10}$$

With these unit vectors, we further define the unit bivectors (relative basis vectors for $e_t = \gamma_0$)

$$\begin{aligned} \sigma_r \equiv& e_r e_t, \\ \sigma_\theta \equiv& \hat{\theta} e_t, \\ \sigma_\phi \equiv& \hat{\phi} e_t. \end{aligned} \tag{11}$$

These bivectors satisfy

$$\sigma_r \sigma_\theta \sigma_\phi = e_t e_r \hat{\theta} \hat{\phi} = i. \tag{12}$$

As in our previous paper [14], we initially attempted to analyze static systems. The set of $\bar{h}$ field that satisfies the spherically symmetric and static matter distribution is assumed to take the form [21,26]

$$\begin{aligned} &\bar{h}(e^t) = f_1 e^t, \quad \bar{h}(e^r) = g_1 e^r + g_2 e^t, \\ &\bar{h}(e^\theta) = e^\theta, \quad\ \ \bar{h}(e^\phi) = e^\phi, \end{aligned} \tag{13}$$

where $f_1$, $g_1$, and $g_2$ are all functions of $r$ only. We could have tried the form $\bar{h}(e^t) = f_1 e^t + f_2 e^r$, but it is more reasonable to set $f_2$ to zero, which is referred to as the 'Newtonian gauge' [26].

The subsequent steps of this study can be summarized as follows:

$$\bar{h}(e^\mu) \xrightarrow{D \wedge \bar{h}(a) = \kappa \bar{h}(a) \cdot S} \omega(a)$$

$$\xrightarrow{\mathcal{R}(a \wedge b) = L_a \omega(b) - L_b \omega(a) + \omega(a) \times \omega(b)} \begin{Bmatrix} \mathcal{R}(a) = \partial_b \cdot \mathcal{R}(b \wedge a) \\ \mathcal{R} = \partial_a \cdot \mathcal{R}(a) \end{Bmatrix}$$

$$\xrightarrow{\mathcal{G}(a) = \kappa \mathcal{T}(a)} g_{\mu\nu} = \underline{h}^{-1}(\mu) \cdot \underline{h}^{-1}(\nu),$$

where the metric $g_{\mu\nu}$ is introduced to compare our results with those predicted by general relativity (GR) and to provide conventional approaches for subsequent applications. Notably, $\omega(a)$ and $\mathcal{R}(a \wedge b)$ can be decomposed into torsion-free components and torsion components [27]. This decomposition allows for the recovery of classical predictions of GR when the impact of torsion is insignificant. Conversely, given the extensive research on these classical predictions available in the existing literature, we can easily incorporate torsion terms into classical results when we deem them to be significant. This allows us to leverage decompositions and enhance our understanding of the phenomena under consideration.

By solving Equation (A154), we can obtain a solution for $\omega(a)$ [26,27], the result is

$$\omega(a) = \omega'(a) + \frac{1}{2}\kappa a \cdot S, \tag{14}$$

where

$$\begin{aligned} \omega'(a) =& (a \cdot e_t G - a \cdot e_r F) e_r e_t - \left(\frac{g_2}{r}\right) a \cdot \hat{\theta} \hat{\theta} e_t \\ &- \left(\frac{g_1 - 1}{r}\right) a \cdot \hat{\theta} e_r \hat{\theta} - \left(\frac{g_2}{r}\right) a \cdot \hat{\phi} \hat{\phi} e_t \\ &- \left(\frac{g_1 - 1}{r}\right) a \cdot \hat{\phi} e_r \hat{\phi} \end{aligned} \tag{15}$$

denotes the torsion-free component of $\omega(a)$, and the new functions $G$ and $F$ are also all functions of $r$ only. From the $\omega$ field, for any bivector $B$, its strength tensor can be obtained directly from Equation (A149) [26,27], the result is

$$\begin{aligned} \mathcal{R}(B) =& \mathcal{R}'(B) + \frac{1}{4}\kappa^2 (B \cdot S) \cdot S \\ &- \frac{1}{2}\kappa (B \cdot D) \cdot S, \end{aligned} \tag{16}$$

where

$$\begin{aligned}\mathcal{R}'(B) =&\alpha_1\sigma_r B\cdot\sigma_r + (\alpha_2\sigma_\theta + \alpha_3 i\sigma_\phi)B\cdot\sigma_\theta \\ &+ (\alpha_2\sigma_\phi - \alpha_3 i\sigma_\theta)B\cdot\sigma_\phi + \alpha_6\sigma_r(B\wedge\sigma_r) \\ &+ (\alpha_4\sigma_\theta - \alpha_5 i\sigma_\phi)(B\wedge\sigma_\theta) \\ &+ (\alpha_4\sigma_\phi + \alpha_5 i\sigma_\theta)(B\wedge\sigma_\phi)\end{aligned} \tag{17}$$

denotes the torsion-free component of $R(B)$, with the understanding that $\sigma_r\wedge\sigma_\theta = \sigma_\phi\wedge\sigma_\theta = 0$, and $\alpha_1,\ldots,\alpha_6$ are given by

$$\begin{aligned}\alpha_1 =&L_r G - L_t F + G^2 - F^2, \\ \alpha_2 =&-L_t\left(\frac{g_2}{r}\right) + \left(\frac{g_1}{r}\right)G - \left(\frac{g_2}{r}\right)^2, \\ \alpha_3 =&L_t\left(\frac{g_1}{r}\right) + \frac{g_1 g_2}{r^2} - \left(\frac{g_2}{r}\right)G, \\ \alpha_4 =&L_r\left(\frac{g_1}{r}\right) + \left(\frac{g_1}{r}\right)^2 - \left(\frac{g_2}{r}\right)F, \\ \alpha_5 =&L_r\left(\frac{g_2}{r}\right) + \frac{g_1 g_2}{r^2} - \left(\frac{g_1}{r}\right)F, \\ \alpha_6 =&(-g_2^2 + g_1^2 - 1)/r^2.\end{aligned} \tag{18}$$

From Equations (16) and (17), the Ricci tensor $\mathcal{R}(a)$ and Ricci scalar $\mathcal{R}$ are given by

$$\begin{aligned}\mathcal{R}(a) =&R'(a) + \frac{1}{2}\kappa^2(a\cdot S)\cdot S \\ &-\frac{1}{2}\kappa a\cdot(D\cdot S), \\ \mathcal{R} =&\mathcal{R}' + \frac{3}{2}\kappa^2 S^2,\end{aligned} \tag{19}$$

where

$$\begin{aligned}\mathcal{R}'(a) =&[(\alpha_1 + 2\alpha_2)a\cdot e_t + 2\alpha_5 a\cdot e_r]e_t \\ &+ [2\alpha_3 a\cdot e_t - (\alpha_1 + 2\alpha_4)a\cdot e_r]e_r \\ &- (\alpha_2 + \alpha_4 + \alpha_6)a\cdot\hat{\theta}\hat{\theta} \\ &- (\alpha_2 + \alpha_4 + \alpha_6)a\cdot\hat{\phi}\hat{\phi},\end{aligned} \tag{20}$$

and

$$\mathcal{R}' = 2\alpha_1 + 4\alpha_2 + 4\alpha_4 + 2\alpha_6, \tag{21}$$

denote the torsion-free components of $\mathcal{R}(a)$ and $\mathcal{R}$, respectively.

In order to solve Einstein Equation (A155), we need to know $T(a)$ given by Equation (A157). Noting that $\psi = \psi(r)$, we find that

$$\begin{aligned}\mathcal{T}(a) =&\langle a\cdot D\psi i\gamma_3\tilde{\psi}\rangle_1 \\ =&\langle a\cdot\bar{h}(e^\mu)\partial_\mu\psi i\gamma_3\tilde{\psi} + \frac{1}{2}\omega(a)\psi i\gamma_3\tilde{\psi}\rangle_1 \\ =&a\cdot(g_1 e^r + g_2 e^t)\langle\partial_r\psi i\gamma_3\tilde{\psi}\rangle_1 + \omega(a)\cdot S.\end{aligned} \tag{22}$$

The terms $\langle\partial_r\psi i\gamma_3\tilde{\psi}\rangle_1$ should be derived from the Dirac Equation (7). Nevertheless, as elaborated in our previous paper [14], the use of this stress–energy tensor prototype can be misleading when applied to gravitational systems consisting of macroscopic bodies. This is because, as $S$ approaches zero, the tensor $T(a)$ defined in (22) also approaches zero. This contradicts our expectation that as $S$ tends to zero, $T(a)$ should represent a tensor describing a classical pressureless ideal fluid.

An appropriate representation of the stress–energy tensor for our specific needs involves decomposing it into two components: one that characterizes the classical pressure-free ideal fluid and another that accounts for quantum effects. By substituting $\nabla$ with $D$, we can rephrase Equation (A113) and express $\mathcal{T}(a)$ as,

$$\mathcal{T}(a) = \rho_m a \cdot e_t e_t - (a \wedge D) \cdot (S \cdot e_t) e_t + [a \cdot D(S \cdot e_t)] \cdot e_t, \tag{23}$$

where we have replaced $v$ with $e_t$ due to the retained gauge freedom to perform arbitrary radial boosts in restricting the $\bar{h}$ function [21,26]. It is important to note that the choice $v = e_t = \gamma_0$ simply fixes the rotation gauge in such a way that the stress–energy tensor takes on the simplest form; there is no other additional physical content. Furthermore, setting $v = e_t$ does not imply that the fluid is at rest or that the observers are comoving with the fluid. In such a setting, all rotation-gauge freedom has been completely removed, as it should be, before one can derive a complete set of physical equations. Note also that $\mathcal{R}(B)$ deals directly with physically measurable quantities, whereas the algebraic structure of the $h$-function is of little direct physical significance. Therefore, the rotation gauge has been fixed by imposing a suitable form for $\mathcal{R}(B)$, rather than restricting the form of $\bar{h}(a)$, as discussed in Refs. [21,26]. Consequently, the proper mass density $\rho_m(r)$, potential energy density $\rho_Q(r)$, and the corresponding mass (energy) $M(r)$ presented later acquire a similar physical meaning as in GR.

The Einstein equation is

$$\mathcal{G}(a) = R(a) - \frac{1}{2} a\mathcal{R} = \kappa \mathcal{T}(a). \tag{24}$$

A direct and efficient approach is to solve the Einstein equation separately for two scenarios: when $a = e_t$ and $a = e_r$. To be specific, in general, our results need to be derived from the following equations:

$$\begin{aligned} e_t \cdot \mathcal{G}(e_t) =& \kappa e_t \cdot \mathcal{T}(e_t), \\ e_r \cdot \mathcal{G}(e_t) =& \kappa e_r \cdot \mathcal{T}(e_t), \\ e_t \cdot \mathcal{G}(e_r) =& \kappa e_t \cdot \mathcal{T}(e_r), \\ e_r \cdot \mathcal{G}(e_r) =& \kappa e_r \cdot \mathcal{T}(e_r). \end{aligned} \tag{25}$$

However, we demonstrate that, for our intended purpose, solving the first two equations listed above is adequate. From (23), we obtain

$$e_t \cdot \mathcal{T}(e_t) = \rho_m - (e_t \wedge D) \cdot (S \cdot e_t). \tag{26}$$

Since $(e_t \wedge D) \cdot (S \cdot e_t) = (e_t \wedge \partial_a) \cdot [a \cdot D(S \cdot e_t)]$, we need to calculate $a \cdot D(S \cdot e_t)$, as follows:

$$\begin{aligned} a \cdot D(S \cdot e_t) =& (a \cdot DS) \cdot e_t + S \cdot (a \cdot De_t) \\ =& [a \cdot \bar{h}(e^r) \partial_r S + \omega(a) \times S] \cdot e_t + S \cdot [\omega(a) \cdot e_t]. \end{aligned} \tag{27}$$

We thus have

$$\begin{aligned} (e_t \wedge \partial_a) \cdot [a \cdot D(S \cdot e_t)] =& (e_t \wedge \partial_a) \cdot [S \cdot (\omega(a) \cdot e_t)] \\ =& e_t \cdot \{[\partial_a \wedge (\omega(a) \cdot e_t)] \cdot S\} \\ =& e_t \cdot \{[-G\sigma_r + \kappa(S \cdot e_t)] \cdot S\} \\ =& \kappa(e_t \cdot S)^2 = \kappa S^2. \end{aligned} \tag{28}$$

Therefore,

$$e_t \cdot \mathcal{T}(e_t) = \rho_m - \kappa S^2. \tag{29}$$

Similarly, we have

$$\begin{aligned} e_r \cdot \mathcal{T}(e_t) =&[e_t \cdot D(S \cdot e_t)] \cdot (e_t \wedge e_r) \\ =&G(S \cdot e_r) \cdot (e_t \wedge e_r) \\ =&0. \end{aligned} \tag{30}$$

On the other hand, for $\mathcal{G}(e_t)$, we find from (19)–(21) that

$$\begin{aligned} \mathcal{G}(e_t) =&R(e_t) - \frac{1}{2}e_t R \\ =&2\alpha_3 e_r - (2\alpha_4 + \alpha_6 + \frac{3}{4}\kappa^2 S^2)e_t \\ &+ \frac{1}{2}\kappa^2(e_t \cdot S) \cdot S - \frac{1}{2}\kappa e_t \cdot (D \cdot S). \end{aligned} \tag{31}$$

Substituting (29)–(31) into $e_t \cdot \mathcal{G}(e_t) = \kappa e_t \cdot \mathcal{T}(e_t)$ and $e_r \cdot \mathcal{G}(e_t) = \kappa e_r \cdot \mathcal{T}(e_t)$, we obtain

$$\begin{array}{rcl} 2\alpha_4 + \alpha_6 &=& -\kappa(\rho_m - \frac{3}{4}\kappa S^2) \\ \alpha_3 &=& 0. \end{array} \tag{32}$$

Interestingly, from (20) we find

$$\partial_a \wedge \mathcal{R}'(a) = 2\alpha_5\sigma_r - 2\alpha_3\sigma_r = 0, \tag{33}$$

which gives

$$\alpha_5 = \alpha_3 = 0. \tag{34}$$

By substituting the expressions of $\alpha$s given by (18) into (32) and (34), our solutions to the Einstein equations can be summarized as follows:

$$\begin{aligned} 2\Big[g_1\partial_r(\frac{g_1}{r}) + (\frac{g_1}{r})^2 - F(\frac{g_2}{r})\Big] - \frac{g_2^2 - g_1^2 + 1}{r^2} =& -\kappa(\rho_m - \frac{3}{4}\kappa S^2), \\ g_2\partial_r(\frac{g_1}{r}) + \frac{g_1 g_2}{r^2} - G\frac{g_2}{r} =&0, \\ g_1\partial_r(\frac{g_2}{r}) + \frac{g_1 g_2}{r^2} - F\frac{g_1}{r} =&0. \end{aligned} \tag{35}$$

The first and third equations in (35) can be combined to give

$$\partial_r\Big[r\Big((g_2)^2 - (g_1)^2 + 1\Big)\Big] = \kappa\Big(\rho_m - \frac{3}{4}\kappa S^2\Big)r^2. \tag{36}$$

Now, if we define

$$M = \frac{r}{2}\Big((g_2)^2 - (g_1)^2 + 1\Big), \tag{37}$$

we find (remembering $\kappa = 8\pi$):

$$M(r) = 4\pi\int_0^r \Big(\rho_m(r') - \frac{3}{4}\kappa S^2(r')\Big)r'^2 dr'. \tag{38}$$

The expression of $M$ strongly implies that it can be identified as the mass (total energy) of a gravitational system within $r$ at any given time $t$. Remarkably, the mass $M$ contains the quantum effects, $-\frac{3}{4}\kappa S^2(r)$, as intended. Naturally, when $S = 0$, the conventional expression for $M$ in general relativity is regained.

As mentioned earlier, there are three functions in the $\bar{h}$ field, namely $f_1$, $g_1$, and $g_2$, that need to be determined by solving Einstein equations. So far, we have obtained expressions for $g_1$ and $g_2$, and they are linked to the observables $\rho_m$ and $S$. Thus, the determination of $f_1$ remains pending. To achieve this, we substitute $a = e^t$ into the torsion Equation (A154), yielding the subsequent equations

$$
\begin{aligned}
D \wedge \bar{h}(e^t) &= \kappa \bar{h}(e^t) \cdot S, \\
\partial_a \wedge \left[a \cdot \bar{h}(e^\mu)\partial_\mu f_1 e^t + f_1 \omega(a) \cdot e^t\right] &= \kappa f_1 e_t \cdot S, \\
g_1 \partial_r f_1 &= -Gf_1, \\
f_1 &= e^{-\int \frac{G}{g_1} dr}.
\end{aligned}
\tag{39}
$$

On the other hand, from the last two equations of (35), it is easy to derive that

$$
G = \partial_r g_1, \quad F = \partial_r g_2. \tag{40}
$$

So, we immediately obtain that

$$
f_1 = 1/g_1. \tag{41}
$$

Now, we turn to the metric tensor $g_{\mu\nu} = g_\mu \cdot g_\nu = \underline{h}^{-1}(e_\mu) \cdot \underline{h}^{-1}(e_\nu)$. From (13) and (41), we obtain

$$
\begin{aligned}
&\underline{h}^{-1}(e_t) = g_1 e_t - g_2 e_r, \quad &&\underline{h}^{-1}(e_r) = \frac{1}{g_1} e_r, \\
&\underline{h}^{-1}(e_\theta) = e_\theta, &&\underline{h}^{-1}(e_\phi) = e_\phi.
\end{aligned}
\tag{42}
$$

So, the metric tensor depends only on $g_1$ and $g_2$. However, the $(g_2)^2$ included in (37) served as the kinematic energy in gravitational systems [26]. To see this, we consider a radially free-falling particle with $v = e_t$. We have

$$
\dot{x} = \frac{dx}{d\tau} = \dot{t}e_t + \dot{r}e_r = \underline{h}(v) = \underline{h}(e_t) = \frac{1}{g_1} e_t + g_2 e_r. \tag{43}
$$

Clearly, in this case, $g_2 = \dot{r}$ represents the radial velocity of the particle. In general, we can understand the physical significance of $g_2$ by rewriting (37) as [26]

$$
\frac{1}{2}(g_2)^2 - \frac{M}{r} = \frac{1}{2}\Big((g_1)^2 - 1\Big), \tag{44}
$$

which is a Bernoulli equation for zero pressure and total non-relativistic energy $\frac{1}{2}((g_1)^2 - 1)$.

But for non-relativistic matter, the contribution of kinematic energy to gravity can be safely disregarded. So, from (37) we have

$$
g_1 = \left(1 - \frac{2M(r)}{r}\right)^{1/2}. \tag{45}
$$

This paper specifically concentrates on non-relativistic matter within static gravitational systems, enabling us to express the metric as follows:

$$
d\tau^2 = \left(1 - \frac{2M(r)}{r}\right) dt^2 - \left(1 - \frac{2M(r)}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{46}
$$

Remarkably, we have obtained a metric that bears a striking resemblance to the familiar form in general relativity. However, it is important to note that this metric was derived from Dirac theory, and the mass $M$ incorporates the quantum effects arising from macroscopic matter.

In this paper, the term $-\frac{3}{4}\kappa S^2(r)$ in the mass–energy expression (38) is referred to as the "quantum potential energy". This concept was first introduced by Bohm [34] in 1952 and later by DeWitt [35] in the same year, with both authors commonly referring to it as the "quantum potential" [29]. Similarly to how traditional mass–energy curves spacetime, quantum potential energy distorts spacetime. This phenomenon is known as "spin-torsion" theory [25,26]. By explicitly decomposing the stress–energy tensor $\mathcal{T}(a)$ into a spin-free part and a spin part, as illustrated in (23), the gravitational strength, represented by the Riemann tensor $\mathcal{R}(B)$, can be explicitly decomposed into a torsion-free component and a

torsion component, as shown in (16). This results in a two-component metric as outlined in (46). Naturally, when $S = 0$, the metric reduces to one that describes a spherically-static gravitational system, as suggested by GR.

It is crucial to grasp the physical significance of the quantum potential energy (QPE) in this metric. To date, in the literature, QPE has been attributed uniquely to microparticles. It has been confirmed that QPE would reduce to a time dilation in spacetime. For instance, when negative muons are captured in atomic s-states, their lifetimes are increased by a time dilation factor corresponding to the Bohr velocity. So, we expect that, if this QPE-induced time dilation is extended to macroscopic matter at cosmic scales, it may be disguised as some extra gravitational potential well. We will show that this extra time dilation can successfully explain the cosmic redshift and galactic rotation curve problem. For microparticles, the troubles encountered all arise from the fact that when the QPE is significant, its gravitational effect is not geometric (i.e., mass-dependent). This results in the concept of the geodesic for test particles becoming ambiguous. Therefore, as mentioned earlier, if we want to apply this metric to gravitational systems composed of macroscopic particles, so that QPE is significant only at cosmic scales and can be neglected locally, we must establish a mechanism that connects the quantum effects of macroscopic matter to the spatial scales of gravitational systems. The mechanism, however, can be clearly demonstrated in the applications of our theory to cosmology and galaxies, as illustrated in the following section.

## 3. Applications: Cosmology and Galaxies

To investigate the mechanism that connects the quantum effects of macroscopic matter to the spatial scales of gravitational systems, we begin by considering a radially symmetric and static gravitational system consisting of $N$ free-falling particles, each with an identical mass of $m$. We assume that the direct collisions and the close encounters due to gravity between the particles can be neglected. This allows us to approximate the gravitational effect of the particles by a smooth distribution of matter. Namely, we assume a smooth mass density function $\rho_m(r)$ and a smooth number density function $\rho(r)$ defined by

$$\rho_m(r) = m\rho(r) = \sum_{i=1}^{N} m\delta(r - r_i). \tag{47}$$

It is evident that the accuracy of the approximation improves as the mass $m$ decreases and the particle number $N$ increases. However, for a fixed mass density $\rho_m(r)$, reducing the mass $m$ leads to an increase in the number density $\rho(r)$. On the other hand, the scale-dependence of a particle's mass in a gravitational system can be easily demonstrated. A particle with a mass of $m$ contributes to the average mass density of a system with a size of $\lambda$ through the following relation:

$$\rho(m, \lambda) \sim \frac{m}{\lambda^3}. \tag{48}$$

Clearly, from the perspective of average mass density, the "effective mass" of a particle decreases as the size of the system increases, following a trend of $\sim\lambda^{-3}$. While this may seem trivial, it becomes crucial in our understanding of quantum effects at cosmic scales. For example, in our Milky Way, $\lambda\sim 10$ kpc. The Sun ($m = M_\odot$) contributes to the average density of the Milky Way as an electron ($m = 9.1 \times 10^{-28} g$) contributes to a system of size $\lambda\sim 230$ cm, a macroscopic size. So, as an assumption, we extend the quantum nature of electrons to macroscopic matter particles distributed in sufficiently large systems.

In a gravitational system, the Dirac equation for a free-falling particle describes a particle of fixed proper mass $m$, a constant spin $\hbar/2$, and a proper probability density $\rho(r)$. We consider the probability density as the number density of the system. In the case of spin-$\frac{1}{2}$ microparticles, the mass $m$ can vary from neutrinos ($m \approx 0$) to neutrons, while the spin remains constant at $\hbar/2$. Therefore, the spin of a particle is independent of its mass. This fact can be naturally extended to macroscopic particles. Typically, when dealing with large masses (macroscopic matter), the negligible quantum effect can be

understood as the short wavelength approximation, commonly referred to as the classical limit. In the context of radially symmetric gravitational systems, we can naturally interpret the negligible quantum effect as a result of the negligibly small number density of particles (since each particle has exactly a fixed value $\hbar/2$ of spin). Given that reducing the mass $m$ results in an increase in the number density $\rho(r)$, we can introduce a decreasing function $m(\lambda)$ ($\lambda$ is the size of the gravitating system), which in turn leads to an increasing function $\rho(r)$. Consequently, the strength of the quantum effect would increase as the distance (or scale) to an observer increases, aligning with our postulation.

To be more precise, we define the density of QPE from (38) as (note that $S^2 = -\frac{1}{4}\rho^2$, and $\rho(r) = \rho_m(r)/m(\lambda)$)

$$\begin{aligned}\rho_Q(r,\lambda) =& -\frac{3}{4}\kappa S^2(r) \\ =&\frac{3}{16}\kappa\rho(r)^2 \\ =&\frac{3}{16}\kappa\left(\frac{\rho_m(r)}{m(\lambda)}\right)^2 \\ =&\frac{\rho_m^2(r)}{\rho_c}\alpha(\lambda),\end{aligned} \tag{49}$$

where $\alpha(\lambda)$ is a dimensionless parameter that depends on the scale $\lambda$. We require that the strength of the QPE vanishes when $\lambda \to 0$ and increases with increasing $\lambda$. Note that, in order to separate the size $\lambda$ dependence of $\rho_Q$ completely from coordinate $r$, we express $\rho_Q(r,\lambda)$ as proportional to $\rho_m^2(r)$, and the constant mass density $\rho_c$ of the entire universe is introduced into the definition of $\rho_Q$ only for dimensional consistency. This choice has nothing to do with our contemplation of the notion that the entire universe may exert a cosmological influence on local galaxies, as elaborated upon in subsequent sections. Needless to say, the functional form of $\alpha(\lambda)$ must be universal, meaning that it is independent of any particular gravitational system, although it can be different for the entire universe and galaxies. The specific properties of $\alpha(\lambda)$ should be determined theoretically from first principles. However, as a first attempt to compare our predicted results with observations, when theoretical formulas for $\alpha(\lambda)$ are lacking, we can derive a phenomenological formula by fitting the data to observations.

From (38), we observe that the expression for $M$ can be rewritten as

$$\begin{aligned}M(r) =& 4\pi\int_0^r(\rho_m(r') + \rho_Q(r',\lambda))r'^2dr' \\ =&4\pi\int_0^r\rho_m(r')r'^2dr' + \frac{4\pi\alpha(r)}{\rho_c}\int_0^r\rho_m^2(r')r'^2dr' \\ =&M_m(r) + M_Q(r),\end{aligned} \tag{50}$$

where we have set $\lambda = r$ in the integrand to reflect the fact that, in $\rho_Q(r',\lambda)$, the scale $\lambda$ is $r$, the size of the "system" with mass $M(r)$. And

$$M_m(r) = 4\pi\int_0^r\rho_m(r')r'^2dr' \tag{51}$$

$$M_Q(r) = \frac{4\pi\alpha(r)}{\rho_c}\int_0^r\rho_m^2(r')r'^2dr' \tag{52}$$

represent the conventional mass (energy) in general relativity and the QPE within $r$, respectively. From (49) and (50), it can be clearly seen that the distribution of QPE exhibits no extra symmetries, i.e., it aligns with the mass distribution. The reason for this is that the scale $\lambda$ characterizes only the amount of QPE for macroscopic matter at cosmic scales. Therefore, as the scale increases, the amount of QPE also increases, causing the quantum effects to vanish locally. This is in contrast to electrons, whose QPE can never vanish at any scale, thus resulting in possible anisotropy if not averaged [31,32].

As previously mentioned, the functional form of $\alpha(\lambda)$ should be determined based on observations. Nonetheless, we have found that the following general form is a suitable choice:

$$\alpha(x) = \frac{A(e^x - 1)}{x^2} + B(e^x - 1) + C, \quad \text{with} \quad x = H_0\lambda, \tag{53}$$

where $H_0$ represents the Hubble constant, which will be defined subsequently. We leave the coefficients $A$, $B$, and $C$, possibly scale dependent, as free parameters to fit the observational data of the gravitational systems under consideration.

### 3.1. Cosmology

Now we are ready to study cosmology. Assuming a static universe with a homogeneous and isotropic matter distribution at large scales, characterized by a constant mass density $\rho_c$, the mass $M$ in (50) then becomes

$$\begin{aligned} M(r) =& 4\pi \int_0^r \rho_c r'^2 dr' + 4\pi\alpha(r) \int_0^r \rho_c r'^2 dr' \\ =& \frac{1}{2} H_0^2 r^3 (1 + \alpha(r)), \end{aligned} \tag{54}$$

where

$$H_0^2 = \frac{8\pi\rho_c}{3} \tag{55}$$

is the Hubble constant. In cosmology, the entire universe can be regarded as an gravitational system with an arbitrarily large scale. To compare observations, we suggest setting the parameters in (53) as $A = 2$, $B = 0$, and $C = -1$. Namely, for cosmology, we assume

$$\alpha_c(\lambda) = \frac{2(e^{H_0\lambda} - 1)}{H_0^2\lambda^2} - 1, \tag{56}$$

which yields desirable results:

$$M(r) = r(e^{H_0 r} - 1) = H_0 r^2 + \frac{1}{2} H_0^2 r^3 + \cdots . \tag{57}$$

Before formulating the metric for the spacetime of the universe, it is important to highlight that the term $\frac{1}{2}H_0^2 r^3$ resulting from $\rho_c$ in $M(r)$ given in (54) should not be disregarded. One could argue that the universe is a distinct gravitational system, exhibiting a homogeneous and isotropic distribution of matter, and infinite in size. Consequently, the observation of gravitational redshift may not be possible under these circumstances. However, this conclusion is derived from Newtonian theory or general relativity, and does not hold true when quantum effects come into play. In fact, the term $\frac{1}{2}H_0^2 r^3$ can be included as the QPE if we choose $C = 0$ in (53). In any case, we should keep in mind that any choice of $\alpha_c(\lambda)$ must be subjected to testing against observations. Moreover, due to the presence of quantum effects at cosmic scales, we have to abandon the idea of the absolute spacetime predicted by general relativity. This is quite similar to the case of special relativity, when Einstein had to relinquish the concept of Newtonian absolute space and time. As a result, the conventional notion of the geometry of the entire universe becomes meaningless. For instance, it loses significance to classify the universe as open, flat, or closed according to its mass density $\rho_c$. Thus, for cosmology, we have from (45)

$$g_1 = \left(1 - \frac{2M(r)}{r}\right)^{1/2} = (3 - 2e^{H_0 r})^{1/2}. \tag{58}$$

Hence, the metric (46) for the universe becomes

$$
\begin{aligned}
d\tau^2 =&(3-2e^{H_0 r})dt^2 - (3-2e^{H_0 r})^{-1}dr^2 \\
&- r^2(d\theta^2 + \sin^2\theta d\phi^2).
\end{aligned} \tag{59}
$$

From this metric, it can be readily deduced that a horizon exists in the static universe, with the distance to any observer in the universe given by

$$
r_h = \frac{1}{H_0}\ln(3/2) \approx \frac{0.4}{H_0}, \tag{60}
$$

which is less than half a Hubble radius $1/H_0$.

As a crucial outcome of our theory, we now derive the Hubble redshift for a static universe model. Suppose a light signal emitted from a source at $r$ and received by an observer at $r=0$, the redshift arising from the time dilation can be readily derived as

$$
1+z = \frac{g_1(0)}{g_1(r)} = \frac{1}{\sqrt{3-2e^{H_0 r}}}. \tag{61}
$$

It is evident that the redshift approaches infinity as the light source approaches the horizon. In the scenario where the source is located near an observer, the redshift can be approximated by a linear function of the distance $r$ when $z \ll 1$, consistent with observations. To see this, we expand the expression in (61) as follows:

$$
z = H_0 r + 2H_0^2 r^2 + \cdots. \tag{62}
$$

This shows that the first term, $H_0 r$, dominates the redshift at low values of $z$. In fact, it is precisely this favorable result that motivated our selection of the function $\alpha_c(\lambda)$ as assumed in (56).

The close sources' redshift $z$ and distance $r$ exhibit a well-established and fundamental observational fact of a linear relationship, one that remains independent of cosmological models. Hence, it is imperative to subject any plausible cosmological model to rigorous testing using relevant observations. We validate our cosmological model by comparing the predicted luminosity distance $d_{\mathrm{L}}$ with the values derived from SN Ia data of the MLCS2k2 Full Sample [36].

The luminosity distance is defined such that the Euclidean inverse-square law for the diminution in light with distance from a point source is preserved. Let $L$ denote the absolute luminosity of a source at distance $r$ and $l$ denote the observed apparent luminosity, and the luminosity distance is defined as

$$
d_{\mathrm{L}} = \left(\frac{L}{4\pi l}\right)^{1/2}. \tag{63}
$$

The area of a spherical surface centered on the source and passing through the observer is just $4\pi r^2$. The photons emitted by the source arrive at this surface having been redshifted by the quantum effect by a factor $1+z$. We therefore find

$$
l = \frac{L}{4\pi r^2}\frac{1}{1+z}, \tag{64}
$$

from which

$$
d_{\mathrm{L}} = r(1+z)^{1/2}. \tag{65}
$$

The coordinate distance $r$ can be expressed in terms of $z$ and $H_0$ from (61), we thus obtain

$$
d_{\mathrm{L}} = \frac{\sqrt{1+z}}{H_0}\ln\left[\frac{1}{2}\left(3 - \frac{1}{(1+z)^2}\right)\right]. \tag{66}
$$

The distance modulus is defined by

$$\mu = m - M = 5\log\left(\frac{d_{\mathrm{L}}}{\mathrm{Mpc}}\right) + 25, \tag{67}$$

where $m$ and $M$ are the apparent magnitude and absolute magnitude related to $l$ and $L$, respectively, and we have explicitly specified that $d_{\mathrm{L}}$ is measured in units of megaparsecs.

We use the full cosmological sample of SNe Ia, MLCS2k2, presented in Table 5 of Riess et al.'s paper [36]. The sample consists of 11 SNe with redshift ranges from 0.01 to 0.1, which is ideal for testing our cosmological model. The only free parameter in our model is the Hubble constant $H_0$, with the best-fitted value being $H_0 = 59.0^{+1.68}_{-1.63}$ km/s/Mpc, as presented in Figure 1.

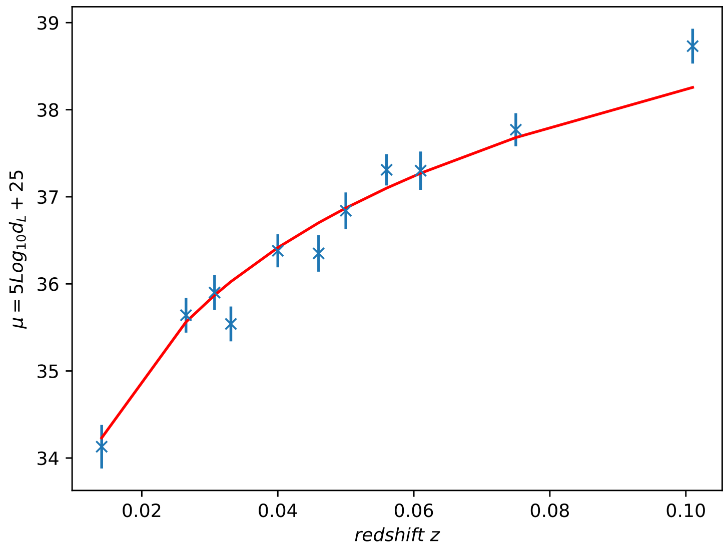


**Figure 1.** The fitting results of the predicted distance moduli given by (67) to the data of the MLCS2k2 full sample [36]. The best fitted value of $H_0$ is $H_0 = 59.0^{+1.68}_{-1.63}$, and $\chi^2 = 0.0018$.

Our cosmological model fits remarkably well with the SNe Ia data, providing evidence that the cosmological redshift may be due to QPE-induced time dilation rather than the expansion of the universe. Our model is based on a static universe, which is made possible by the assumption of quantum effects at cosmic scales, as described by QPE, acting as a repulsive force to balance gravity. When Einstein introduced the cosmological constant $\Lambda$ to balance gravity and obtain a static universe model, he failed, because $\Lambda$, regardless of its physical significance, is independent of the matter distribution. Consequently, any perturbations in matter result in unstable solutions of Einstein's equations. In contrast, according to our proposal, QPE is not independent of matter distribution but is determined by the matter density, as shown in Equation (49). As a result, any perturbations in matter distribution result in corresponding perturbations in QPE, to balance the extra gravity, allowing the universe to remain static.

It is worth noting from (49) that QPE depends on two independent factors: $\rho_m^2(r)$ and $\alpha(\lambda)$. As mentioned earlier, it is essential for $\alpha(\lambda)$ to satisfy the condition that $\alpha(\lambda)$ approaches zero as $\lambda$ approaches zero. Thus, even though galaxies possess a high but finite central matter density, the QPE would vanish in that region. However, in the context of cosmology, the QPE that is dependent on $\lambda$ displays some intricate yet justifiable characteristics. To see this, let us rephrase the function $\alpha_c(\lambda)$ in (56) as follows:

$$\alpha_c(\lambda) = \frac{2}{H_0\lambda} + \frac{1}{3}H_0\lambda + \cdots. \tag{68}$$

It is evident that the first term $2/H_0\lambda$ plays a crucial role in generating a linear correlation between redshift and distance at a low redshift. At first sight, the inverse relationship between this term and $\alpha_c(\lambda)$ may appear to contradict the requirement that $\alpha(\lambda)$ approaches zero as $\lambda$ approaches zero. However, upon further examination, this does not turn out to be the case. To grasp this paradox, it is essential to recognize that it is $M_Q(\lambda)$, not $\rho_Q(r,\lambda)$, that determines the metric and thus the geometry of spacetime for a specific observer, as shown in (59).

In cosmology, the redshift of light signals emitted from distant sources is entirely attributed to the QPE, which varies with distance. However, this distance-dependent phenomenon does not imply any preferred direction in the universe, as demonstrated both theoretically and observationally.

Although we prefer the presented static universe model, which lacks expansion, a big bang, and evolution of the entire universe, all local gravitational systems, such as galaxies and clusters of galaxies, can still evolve. In particular, for macroscopic systems, such as the solar system, planetary systems, and black holes, the evolution will continue to be governed by general relativity as usual.

### 3.2. Our Solution to the Galactic Rotation Curve Problem

After establishing the geometry of the entire universe's spacetime, we now shift our focus to examining the galaxies. Traditionally, based on Newtonian theory, when a stellar system achieves a state of equilibrium, the kinetic energy supports the gravity, and this equilibrium state must adhere to the virial theorem. However, astronomical observations indicate that the gravitational force exerted by ordinary matter is insufficient to counterbalance the kinetic energy of the system. As a result, the presence of a mysterious substance known as dark matter has been postulated to account for the discrepancy in mass, often referred to as the missing mass problem.

Based on the theory presented in this paper, we can propose an alternative explanation that challenges the need for dark matter. In fact, in the mass expression provided in Equation (38), the term QPE defined by Equation (49) serves as the counterpart to dark matter.

The mass expression presented in Equation (50) is universally applicable to any radially symmetric system, making it suitable for investigating spherical galaxies. As mentioned earlier, $\alpha(\lambda)$ should also be a universal function of scale $\lambda$. We assume a functional form given by (53). In the context of cosmology, the scale can be regarded as arbitrarily large up to $r_h$, and the reduced specific form of $\alpha_c(\lambda)$ provided in Equation (56) has been demonstrated to align with observational data. For galaxies, we use the parameters in (53) with $A = 0$ and $C = 0$ to obtain

$$\alpha_g(\lambda) = B(e^{H_0\lambda} - 1), \tag{69}$$

where, in this paper, we consider $B$ as a constant, although it may be a parameter, possibly scale-dependent, to be determined by observations. When studying galaxies, the scale $\lambda$ of interest is much smaller than the Hubble radius $1/H_0$, thus $\alpha_g(\lambda)$ can be approximated by $\alpha_g(\lambda) = BH_0\lambda$.

It is essential to account for the impact of the entire universe on local galaxies of finite mass, which is commonly referred to as the boundary condition when the metric of spacetime is concerned. Specifically, we mandate that the metric surrounding a local galaxy adheres to the condition that, as $r$ approaches infinity, it converges to the metric of the entire universe (59), instead of the metric for flat spacetime, which has been conventionally used. To achieve this, we replace $\rho_m(r)$ in Equation (50) with $\rho_m(r) + \rho_c$. Since the impact of cosmological effects on local galaxies is unclear, we make the assumption that

$$\alpha_c(\lambda) = \frac{A(\lambda)(e^{H_0\lambda} - 1)}{H_0^2\lambda^2} - 1 \tag{70}$$

when considering $\rho_c$, allowing us to determine the function $A(\lambda)$ based on observations of galaxies. We thus have from (50)

$$\begin{aligned} M(r) =& 4\pi \int_0^r (\rho_m(r') + \rho_c) r'^2 dr' + 4\pi\alpha_c(r) \int_0^r \rho_c r'^2 dr' + \frac{4\pi\alpha_g(r)}{\rho_c} \int_0^r \rho_m^2(r') r'^2 dr' \\ =& M_b(r) + M_m(r) + M_Q(r), \end{aligned} \tag{71}$$

where

$$\begin{aligned} M_b(r) =& \frac{A(r)}{2}(e^{H_0 r} - 1) r, \\ M_m(r) =& 4\pi \int_0^r \rho_m(r') r'^2 dr', \\ M_Q(r) =& \frac{B4\pi(e^{H_0 r} - 1)}{\rho_c} \int_0^r \rho_m^2(r') r'^2 dr' \\ =& \frac{B32\pi^2(e^{H_0 r} - 1)}{3H_0^2} \int_0^r \rho_m^2(r') r'^2 dr'. \end{aligned} \tag{72}$$

Evidently, $M_b(r)$ signifies the contribution of the quantum effects of the background mass density of the universe, $M_m(r)$ denotes the conventional mass, and $M_Q(r)$ represents the quantum effects of a galaxy itself. We thus have

$$\begin{aligned} g_1 =& \left[1 - \frac{2(M_b(r) + M_m(r) + M_Q(r))}{r}\right]^{1/2} \\ =& \left[1 - A(r)(e^{H_0 r} - 1) - \frac{2M_m(r)}{r} - \frac{2M_Q(r)}{r}\right]^{1/2}. \end{aligned} \tag{73}$$

We require $g_1$ for galaxies to satisfy the boundary condition, i.e., when $r \to \infty$, $g_1 \to (3 - 2e^{H_0 r})^{1/2}$, the value for cosmology as shown in (58). For galaxies of finite mass or finite size, the last two terms $\frac{2M_m(r)}{r} + \frac{2M_Q(r)}{r} \to 0$ when $r \to \infty$. We thus require $A(r) \to 2$ when $r \to \infty$. This condition imposes a nature constraint on the cosmological effect on local galaxies, as shown subsequently.

Our findings indicate that, from the perspective of mass–energy, the contributions of QPE to the metric exhibit no observable distinctions from conventional mass. Both QPE and conventional mass contribute to time dilation and distance contraction in precisely an identical manner, as demonstrated in Equation (46). As such, it is reasonable to employ the conventional approach from general relativity when considering the stress–energy tensor and geometry of spacetime.

Although our findings are presented in relativistic form, the transition to a non-relativistic scenario for galaxies is straightforward. Actually, we can directly utilize the mass expression provided in Equation (71) and apply Newtonian theory to investigate galaxies.

Obviously, near the center of a galaxy, the gravitational effects arising from $M_b(r)$ and $M_Q(r)$ can be neglected when compared to those of $M_m(r)$. Towards the outer regions, both $M_b(r)$ and $M_Q(r)$ become significant, while the contribution of $M_m(r)$ diminishes with increasing radius $r$.

It is interesting to investigate the circular velocity of a test particle around a galaxy, which can be approximated by

$$\begin{aligned} v^2(r) =& \frac{M(r)}{r} \\ =& \frac{A(r)}{2} H_0 r + \frac{4\pi}{r} \int_0^r \rho_m(r') r'^2 dr' + \frac{B32\pi^2}{3H_0} \int_0^r \rho_m^2(r') r'^2 dr', \end{aligned} \tag{74}$$

where we have neglected the terms with $H_0^n r^n$ when $n \geq 2$ for the scale of galaxies. For a galaxy with a finite size $a$, when $r$ increases to the outer regions with $r \gg a$, we have

$$v(r) = \sqrt{\frac{A(r)}{2} H_0 r + M_Q(a)/a}. \tag{75}$$

This expression does not imply a constant value of $v(r)$ for large radii, as the $H_0 r$ term in (75) becomes more and more significant with increasing $r$. In particular, this leads to a universal centrifugal acceleration (recall that $A(r) \to 2$ for $r \to \infty$)

$$\frac{v^2}{r} = cH_0, \tag{76}$$

where $c$ is the speed of light. This result reflects the fact that the matter in the entire universe can have a local observable effect in galaxies, a fact first discovered in MOND theory but that cannot be explained within the theory itself [10,37]. In fact, our cosmological model yields the value of $H_0 = 59.0$, resulting in $cH_0 = 5.74 \times 10^{-8} \text{cm/s}^2$. A value which is close to, but obviously larger, than that found for the universal acceleration parameter $a_0 = 1.2 \times 10^{-8}$ cm/s$^2$ of the MOND theory [9,38]. This fact has a natural explanation. As an increasing function of $r$, $A(r) \to 2$ only at cosmological distances, typically hundreds of Mpc. The value of $a_0$ is smaller than $cH_0$ only because it is detected at distances in the outer regions of galaxies, which are much smaller than the cosmological distance. Notably, conformal gravity yields comparable results when fitting rotation curve data [13,39]. In our theory, these phenomena are not considered mysteries. The cosmological effects on local galaxies are the natural results of the boundary condition, which is determined by the requirement that the spacetime metric around an isolated galaxy, created by the mass (energy) specified in (72), should align with that of the entire universe, as demonstrated in (59), when $r \to \infty$.

Despite these remarkable achievements, the most effective way to validate our theory is by leveraging the abundant observational data on the rotation curves of flattened dwarf and spiral galaxies. As our results were obtained for spherical systems, they must be converted into axisymmetric systems where cylindrical coordinates are more suitable. One could replicate the procedure for axisymmetric systems [21,40], similarly to what was performed for spherical systems in this paper. However, a more efficient approach to achieve our objectives is to consider the relevant terms on the right-hand side of Equation (74) as the gravitational potential generated by a point mass (or a mass element). Subsequently, we calculate the total potential for disk galaxies in cylindrical systems $(R, \phi, z)$ [13].

The first term on the right-hand side of (74) is $A(r)cH_0 r/2$. This is a linear potential originating from the quantum effect of the entire universe, and thus is independent of specific local galaxies. We simply express the contribution of this term to the total circular velocity of a test particle on a thin-disk as

$$v_L^2(R) = A(R)cH_0 R. \tag{77}$$

In this paper, we initially overlook the gradually increasing nature of the function $A(R)$ and consider it as a constant $A_0$, which can be determined using rotation curve data.

The second term on the right-hand side of (74) is the usual Newtonian potential. We write the corresponding potential in cylindrical coordinates as

$$\Phi_m(R,z) = -G \int_0^\infty dR' \int_0^{2\pi} d\phi' \int_{-\infty}^\infty dz' \frac{R' \rho_m(R', z')}{(R^2 + R'^2 - 2RR' \cos\phi' + (z - z')^2)^{1/2}}, \tag{78}$$

where $G$ is the gravitational constant (we temporarily transition back from natural units from now on in this subsection). Inserting the cylindrical coordinate Green's function Bessel function expansion

$$\frac{1}{(R^2 + R'^2 - 2RR'\cos\phi' + (z-z')^2)^{1/2}} = \sum_{-\infty}^{\infty} \int_0^\infty dk J_m(kR) J_m(kR') e^{im(\phi-\phi') - k(|z-z'|)} \quad (79)$$

into (78) yields

$$\Phi_m(R,z) = -2\pi G \int_0^\infty dk \int_0^\infty dR' \int_{-\infty}^\infty dz' R' \rho_m(R',z') J_0(kR) J_0(kR') e^{-k|z-z'|}. \quad (80)$$

For razor-thin exponential disks with $\rho_m(R,z) = \Sigma(R)\delta(z) = \Sigma_0 e^{-R/R_0}\delta(z)$, where $\Sigma_0$ is the central surface mass density and $R_0$ is the disk scale length, the potential is

$$\Phi_m(R) = -\pi G \Sigma_0 R_0 y [I_0(y/2)K_1(y/2) - I_1(y/2)K_0(y/2)], \quad (81)$$

where $y \equiv \frac{R}{R_0}$, $I_n$ and $K_n$ are modified Bessel functions. If we differentiate Equation (81) with respect to $R$, we obtain the circular velocity contributed by the Newtonian potential

$$v_m^2(y) = R\frac{\partial \Phi_m}{\partial R} = \frac{\pi c^2}{R_0}\left(\frac{GM_\odot}{c^2}\right)\left(\frac{\Sigma_0}{M_\odot}\right) R_0^2 y^2 [I_0(y/2)K_0(y/2) - I_1(y/2)K_1(y/2)]. \quad (82)$$

Another potential of the quantum effect for galaxies on the right-hand side of (74) is $(64\pi^2/3H_0)\int_0^r \rho_m^2(r')r'^2 dr'$. This corresponds to a logarithmic potential, and the circular velocity for the razor-thin exponential disk is

$$\begin{aligned} v_Q^2(R) =& \frac{B 8\pi G^2}{3cH_0 R_0} \int_0^R dR' \int_0^{2\pi} d\phi' \int_{-\infty}^\infty dz' R' \Sigma^2(R')\delta(z') \\ =& \frac{B 4\pi^2 G^2 \Sigma_0^2 R_0}{3cH_0}\left[1 - \left(1 + \frac{2R}{R_0}\right)e^{-2R/R_0}\right]. \end{aligned} \quad (83)$$

The total circular velocity used to fit the data is therefore

$$v(R) = \sqrt{v_L^2(R) + v_m^2(R) + v_Q^2(R)}. \quad (84)$$

We fit the predicted circular velocity to the data provided in the Spitzer Photometry and Accurate Rotation Curves (SPARC) database [41]. The SPARC database is the largest sample, to date, of rotation curves for every galaxy. It is a sample of 175 nearby galaxies with new surface photometry at 3.6 mu, and high-quality rotation curves from previous HI/H studies. As a first try, we consider each of the 175 galaxies in the sample as a razor-thin exponential disk characterized by a central surface mass density $\Sigma_0$ and a disk scale length $R_0$. In addition to $\Sigma_0$ and $R_0$, we treat $A(r) = A_0$ in (77) and $B$ in (83) as free parameters in our fitting. Figure 2 displays 12 of them, including 6 low-surface-brightness (LSB) galaxies (6 upper panels) and 6 high-surface-brightness (HSB) galaxies (6 lower panels).

The fittings reveal that $A_0 = 0.02$ for all galaxies, with $B = 3.0$ for LSB galaxies and $B = 0.3$ for HSB galaxies. The cosmological effect on local galaxies thus suggests an acceleration of $A_0 cH_0 \sim 10^{-9}$ cm/s$^2$, significantly smaller than the universal value of $cH_0 = 5.74 \times 10^{-8}$ cm/s$^2$. Our fittings further suggest that $A_0$ can be substituted with

$$A(R) = \frac{A_0 + 2(R/\mathrm{Mpc})^2}{1 + (R/\mathrm{Mpc})^2}. \quad (85)$$

This slowly varying function provides a good fit to the rotation curve data and also fulfills the requirement that $A(R)$ converges to 2 as $R$ approaches cosmological distances. On the other hand, the value of B for LSB galaxies exceeds that for HSB galaxies, suggesting that LSB galaxies require relatively more QPE than HSB galaxies, and $B(\lambda)$ is also scale dependent.

All these results indicate that, in terms of mass-energy, scale-dependent quantum phenomena do not exhibit a linear superposition, but rather a hierarchical process.

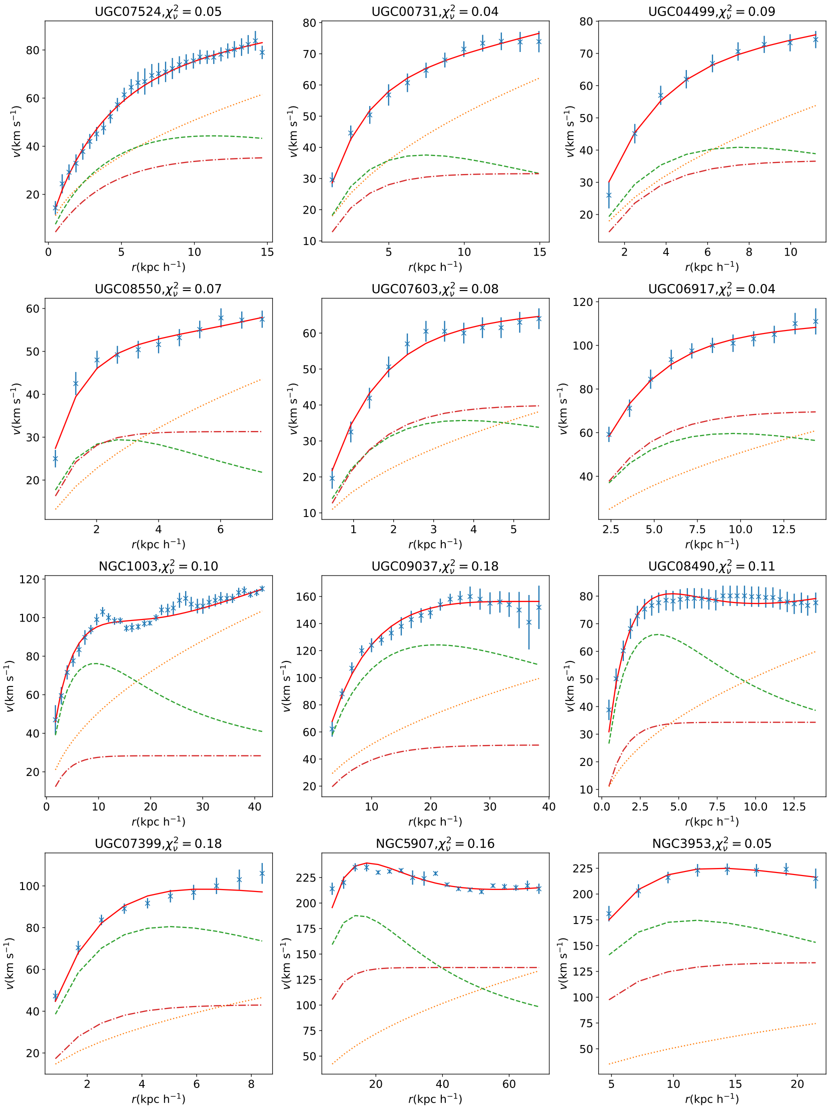


**Figure 2.** The best-fitting rotation curves of LSB (upper 6 panels) and HSB (lower 6 panels) galaxies with the model presented in this paper. The panels are listed by the increasing effective surface brightness of the galaxies. In each panel, the dotted curve shows the contribution from the cosmological quantum effect, given by (77); the dashed curve indicates the contribution from the luminous Newtonian potential given by (82); the dash-dotted curve shows the contribution from the quantum effect of the galaxies themselves given by (83); and finally, the solid curve is the total circular velocity given by (84).

Despite the very simple model used to describe the matter distribution in LSB and HSB galaxies, the fittings can still capture the main phenomenological results observed in other theories of gravity. For instance, the quantum effect contributions ($v_L^2 + V_Q^2$) dominate most regions of LSB galaxies, whereas this is only true for the outer regions of HSB galaxies.

Our proposed quantum effect on large scales has proven to be successful in explaining the mass discrepancy problem in galaxies.

## 4. Conclusions and Discussions

We have proposed the quantum nature of spacetime at cosmic scales. To investigate this postulation, we examined spherically symmetric and static gravitational systems composed of free-falling particles with identical mass $m$. We assumed that the matter distribution can be smoothed using the proper number density $\rho(r)$ and proper mass density $\rho_m(r)$. We further assumed that the Dirac spinor $\psi(r)$ can fully capture all the physical

aspects of the particles and satisfies the covariant Dirac equation. Therefore, we recognize $\rho(r) = \psi(r)\tilde{\psi(r)}$ and $\rho_m(r) = m\psi(r)\tilde{\psi(r)} = m\rho(r)$. For a given mass density $\rho_m(r)$, the interplay between the number density $\rho(r)$ and the particle mass $m$ offers us a inherent mechanism to establish a connection between the spin density $S(r) = \frac{\hbar}{2}\psi(r)i\gamma_3\tilde{\psi(r)}$ and the scale $\lambda$ of gravitational systems.

We utilized gauge theory gravity (GTG) [26,27] to achieve the subsequent derivations. After solving Einstein equations, the metric of the spacetime for the radially symmetric and static gravitational systems was derived, as shown in (46). The fundamental results obtained thus far require additional physical interpretations and practical applications. We found that the quantum potential energy (QPE) defined by (49)

$$\rho_Q(r,\lambda) = -\frac{3}{4}\kappa S^2(r) = \rho_m^2(r)\alpha(\lambda)/\rho_c$$

is contained in the metric via the mass (energy) $M(r)$, where $\alpha(\lambda)$ is a dimensionless function of the scale of the system involved. The mass within radius $r$ can then be expressed in (50).

It is important to note that at each radius $r$, we consider $M(r)$ as the mass of a gravitational system; therefore, we identify $r$ as the scale $\lambda$ of the "system". As such, for any given mass density $\rho_m(r)$ of a gravitational system, we can derive the metric that encompasses not only the classical mass but also the quantum potential energy (QPE). When the scale of the gravitational systems are macroscopic, for instance, the solar system, the QPE can be neglected, and the Einstein–Newtonian theory of gravity is recovered.

The dimensionless function $\alpha(\lambda)$ is universal, and it should be determined theoretically from first principles. However, when theoretical formulas are lacking, we can derive a phenomenological formula by fitting data to observations. We discovered that the general expression presented in (53) successfully achieved our objective.

While our fundamental results were initially derived for radially symmetric and static systems, they can be applied to any static gravitational system. By substituting the proper mass density $\rho_m$ with $\rho_m + \rho_Q + +\rho_c + \rho_c\alpha_c(\lambda)$, where $\rho_c$ is the constant mass density of the universe, we establish a fundamental boundary condition that the spacetime metric encompassing any local gravitational system must adhere to. Remarkably, this boundary condition inherently gives rise to the cosmological impact on local galaxies, as evidenced by observations of galactic rotation curves [9,10,13,38,41,42].

When applied to cosmology, our model yields a static universe. The predicted luminosity distance–redshift relation fit remarkably well with SNe Ia data, providing evidence that the cosmological redshift may be due to QPE-induced time dilation rather than expansion of the universe. It is not surprising that when we extend Dirac's theory for free electrons to macroscopic particles (including stars within galaxies and galaxies within the entire universe), we are effectively moving away from the notion of a preferred absolute spacetime as suggested by general relativity and commonly accepted by many physicists, as illustrated in the standard ΛCDM cosmology. Without the existence of absolute spacetime, physical processes should be described from the perspective of any observer in the universe independently and equivalently, without relying on the absolute spacetime reference frame of the universe. Therefore, the state of a celestial body should be characterized in the frame associated with a specific observer rather than in the frame of the absolute spacetime of the universe. Consequently, we assume that, for any observer in the universe, a distant celestial body behaves akin to a micro-particle and can be effectively described using Dirac's theory. On the contrary, in a scenario where absolute spacetime exists, all matter particles, including observers, must be described in relation to it. As a result, the motion of distant celestial bodies is typically explained by a combination of the local velocity and the overall expansion velocity. In this context, we should not anticipate quantum effects at large scales for any observers.

When considering galaxies, it is intriguing to note that quantum effects can serve as a substitute for dark matter. To validate our theory, the most effective approach was to

utilize the extensive observational data on the rotation curves of dwarf and spiral galaxies. Remarkably, our proposed quantum effect at large scales successfully addressed the mass discrepancy issue in galaxies. Specifically, our theory presented a fitting formula, as outlined in (85), which elegantly explains the correlation between the universal acceleration $cH_0$ suggested by cosmology, $a_0$ proposed by MOND, and $0.02cH_0$ resulting from the cosmological effect on local galaxies [13].

Although not investigated in the present study, it is interesting to note that there is another very important piece of evidence supporting our new theory about the quantum effects of macroscopic matter at cosmic scales. This evidence comes from a large-scale structural survey of the universe, which reveals the fractal geometry of matter distribution at these scales, as cited in Ref. [43]. It has now been established that the quantum randomness of the electron can be mimicked by Brownian motion; the simulated trajectory is continuous but non-differentiable. In fact, the non-differentiable trajectory or path of an electron exhibits a fractal structure due to the uncertainty principle between its position and momentum in the conventional matrix and tensor versions of quantum mechanics [44]. Naturally, the observations of the fractal geometry of the large-scale structure of the universe can be regarded as independent and compelling evidence that supports our assumption when extending the reasoning about the relationship between the quantum nature of matter and its fractal path from microscopic matter to macroscopic matter.

In conclusion, we have expanded the quantum characteristics of microparticles to encompass macroscopic matter at cosmic scales. As illustrated, this expansion was remarkably coherent. The subsequent outcomes were derived within the framework of the firmly established theory of gravity (GTG) and mathematical language (STA). While there is room for improvement in the technical details of our derivations, the fundamental discoveries are credible and, notably, were verified through astronomical observations of SNe Ia and galactic rotation curve data. In particular, our proposal provides an alternative view point to understand the quantum nature of spacetime.

The manifestation of physical laws should be independent of the mathematical language used, and thus we believe that our proposal could also be accomplished, for instance, through the tensor analysis approach. Of course, using an unfamiliar language may make it challenging for readers to comprehend our theory. However, it has been shown that STA and GC can significantly streamline the calculations in our area of concern. Additionally, STA can illustrate the cohesive link between classical mechanics and quantum mechanics, thereby enhancing our comprehension of the enigmatic aspects of quantum theory and spacetime.

For future work, we could test our theory through observations of gravitational lensing, cosmic microwave background radiation, and more sophisticated models for rotation curves. In particular, we could investigate the bullet cluster problem, which is considered the best evidence for the existence of dark matter. A more ambitious project would be to derive the $\alpha(\lambda)$ formula from first principles.

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**Appendix A. Spacetime Algebra and Geometric Calculus**

Spacetime algebra (STA) is generated by a 4-dimensional Minkowski vector space. The inner and outer products of the four orthonormal basis vectors $\{\gamma_\mu, \mu = 0\ldots3\}$ are defined to be

$$\begin{aligned}\gamma_\mu \cdot \gamma_\nu \equiv \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) &\equiv \eta_{\mu\nu} = \mathrm{diag}(+ - - -) \\ \gamma_\mu \wedge \gamma_\nu &\equiv \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu).\end{aligned} \tag{A1}$$

A full basis for the STA is

$$1,\ \{\gamma_\mu\},\ \{\sigma_k, i\sigma_k\},\ \{i\gamma_\mu\},\ i \tag{A2}$$

where $\sigma_k \equiv \gamma_k\gamma_0, k = 1\ldots3$ constitute the orthonormal basis of a 3-dimensional vector space relative to $\gamma_0$, and $i = \gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_1\sigma_2\sigma_3$ is called pseudoscalar. The STA is a linear space of dimensions 16. We refer to the general elements of STA as multivectors, and each multivector decomposes into a sum of elements of different grades. If a multivector contains only grade-$r$ components, we call it homogeneous, and is denoted by $A_r$. We call grade-0 multivectors scalars, grade-1 vectors, grade-2 bivectors and grade-3 trivectors. A grade-$r$ multivector is called simple or an $r$-blade if and only if it can be written as

$$A_r = a_1 \wedge a_2 \wedge \ldots \wedge a_r = \frac{1}{r!}\sum_\pi (\mathrm{sgn}\pi) a_{\pi(1)}a_{\pi(2)}\ldots a_{\pi(r)}, \tag{A3}$$

where $\pi$ is a permutation of 1 through $r$, 'sgn $\pi$' is the sign of the permutation, 1 for even and $-1$ for odd, and the sum is over all $r!$ possible permutations. For $r = 2$, this reduces the outer product of two vectors

$$a \wedge b = \frac{1}{2}(ab - ba), \tag{A4}$$

which is in agreement with the definition in (A1), and clearly $a \wedge a = 0$.

The geometric product of a grade-$r$ multivector $A_r$ with a grade-$s$ multivector $B_s$ is defined simply by $A_rB_s$, which decomposes into

$$A_rB_s = \langle A_rB_s\rangle_{|r-s|} + \langle A_rB_s\rangle_{|r-s|+2} + \ldots + \langle A_rB_s\rangle_{r+s}, \tag{A5}$$

where $\langle X\rangle_r$ denotes the projection onto the grade-$r$ part of $X$. The grade-0 (scalar) part of $X$ is written as $\langle X\rangle$. We employ "$\cdot$" and "$\wedge$" symbols to denote the lowest-grade and highest-grade terms in (A5), so that

$$A_r \cdot B_s = \langle A_rB_s\rangle_{|r-s|}, \tag{A6}$$

$$A_r \wedge B_s = \langle A_rB_s\rangle_{r+s}, \tag{A7}$$

which are called inner and outer products, respectively. They represent a generalization of the geometric product between two vectors $a$ and $b$, which can be expressed as follows:

$$ab = a \cdot b + a \wedge b. \tag{A8}$$

We define the reverse of a geometric product $AB$ by $(AB)^\sim = \tilde{B}\tilde{A}$, so that for vectors $a_1, a_2, \ldots, a_r$, we have

$$(a_1a_2\ldots a_r)^\sim = a_ra_{r-1}\ldots a_1. \tag{A9}$$

This reverse operator obeys the rule

$$(A+B)^\sim = \tilde{A} + \tilde{B}, \tag{A10}$$

$$\langle\tilde{A}\rangle = \langle A\rangle, \tag{A11}$$

$$\tilde{a} = a, \tag{A12}$$

where $a$ is a vector. It is easy to show that

$$\tilde{A}_r = (-1)^{r(r-1)/2} A_r. \tag{A13}$$

Thus, suppose $r \leq s$, the inner and outer products satisfy the symmetry properties

$$\begin{aligned} A_r \cdot B_s &= (-1)^{r(s-1)} B_s \cdot A_r, && \text{(A14)} \\ A_r \wedge B_s &= (-1)^{rs} B_s \wedge A_r. && \text{(A15)} \end{aligned}$$

The scalar product is defined by

$$A * B = \langle AB \rangle. \tag{A16}$$

From (A6), $A_r * B_s$ is nonzero only if $r = s$, thus the scalar product (A16) is commutative

$$\langle AB \rangle = \langle BA \rangle, \tag{A17}$$

which proves to be very useful in our calculations. The inner product and outer product obey the distributive rule

$$\begin{aligned} A \cdot (B + C) = &= A \cdot B + A \cdot C && \text{(A18)} \\ A \wedge (B + C) &= A \wedge B + A \wedge C. && \text{(A19)} \end{aligned}$$

The outer product is associative

$$A \wedge (B \wedge C) = (A \wedge B) \wedge C. \tag{A20}$$

The inner product is not associative, but homogeneous multivectors obey

$$\begin{aligned} A_r \cdot (B_s \cdot C_t) &= (A_r \wedge B_s) \cdot C_t, \ \text{ for } \ r + s \leq t, && \text{(A21)} \\ A_r \cdot (B_s \cdot C_t) &= (A_r \cdot B_s) \cdot C_t, \ \text{ for } \ r + t \leq s. && \text{(A22)} \end{aligned}$$

For any vector $a$, it can be proved that

$$\begin{aligned} a \cdot A_r = \langle a A_r \rangle_{r-1} &= \frac{1}{2}(a A_r - (-1)^r A_r a), && \text{(A23)} \\ a \wedge A_r = \langle a A_r \rangle_{r+1} &= \frac{1}{2}(a A_r + (-1)^r A_r a). && \text{(A24)} \end{aligned}$$

From which, we immediately have

$$a A_r = a \cdot A_r + a \wedge A_r. \tag{A25}$$

We also have the following useful identities

$$\begin{aligned} a \cdot (A_r B) &= (a \cdot A_r) B + (-1)^r A_r (a \cdot B) && \text{(A26a)} \\ &= (a \wedge A_r) B - (-1)^r A_r (a \wedge B) && \text{(A26b)} \\ a \wedge (A_r B) &= (a \wedge A_r) B - (-1)^r A_r (a \cdot B) && \text{(A26c)} \\ &= (a \cdot A_r) B + (-1)^r A_r (a \wedge B). && \text{(A26d)} \end{aligned}$$

These identities imply the following particularly useful relations:

$$\begin{aligned} a \cdot (A_r i) &= (a \wedge A_r) i, && \text{(A27)} \\ a \wedge (A_r i) &= (a \cdot A_r) i, && \text{(A28)} \end{aligned}$$

where $i$ is a pseudoscalar, and we have used the fact that $a \wedge i = 0$.

Any vector $a$ can be decomposed in terms of $\{\gamma_\mu\}$ into

$$a = a \cdot \gamma_\mu \gamma^\mu = a \cdot \gamma^\mu \gamma_\mu, \tag{A29}$$

where the summation convention is implied. Similarly, an arbitrary multivector $A$ can be decomposed into

$$A = \sum_{\mu<\ldots<\nu} A_{\mu\ldots\nu}\gamma^{\mu} \wedge \ldots \wedge \gamma^{\nu}, \tag{A30}$$

where

$$A_{\mu\ldots\nu} = A \cdot (\gamma_{\nu} \wedge \ldots \wedge \gamma_{\mu}). \tag{A31}$$

We further define the commutator product

$$A \times B = \frac{1}{2}(AB - BA), \tag{A32}$$

which satisfies the Jacobi identity

$$A \times (B \times C) + B \times (C \times A) + C \times (A \times B) = 0. \tag{A33}$$

In addition, we have the Leibnitz rule for the commutator product

$$A \times (BC) = (A \times B)C + B(A \times C). \tag{A34}$$

This rule is particular useful with bivectors. It can be proved that

$$A_2 \times A_r = \frac{1}{2}(A_2 A_r - A_r A_2) = \langle A_2 A_r \rangle_r, \tag{A35}$$

which means that if one of the factors is a bivector, then the commutator product preserves its grade. Thus, for $\langle A \rangle_1 = 0$, we have

$$A_2 A = A_2 \cdot A + A_2 \times A + A_2 \wedge A. \tag{A36}$$

Since the commutator product with a bivector is grade preserving, the identity in (A34) still holds if $A = A_2$ and we replace all geometric products with either inner or outer products:

$$\begin{aligned} A_2 \times (B \cdot C) &= (A_2 \times B) \cdot C + B \cdot (A_2 \times C), \\ A_2 \times (B \wedge C) &= (A_2 \times B) \wedge C + B \wedge (A_2 \times C). \end{aligned} \tag{A37}$$

Finally, for any vector $a$ and trivector $T$, it can be proved that

$$(a \cdot T) \times T = 0. \tag{A38}$$

Geometric calculus (GC) is the extension of a geometric algebra (like STA) to include differentiation and integration. Let the multivector $F$ be an arbitrary function of a multivector argument $X$, then the derivative of $F(X)$ with respective to $X$ in the $A$ direction is defined by

$$A * \partial_X F(X) \equiv \lim_{\tau \to 0} \frac{F(X + \tau A) - F(X)}{\tau}, \tag{A39}$$

where the multivector partial derivative $\partial_X$ inherits the multivector properties of its argument $X$. Suppose that the set $\{e_k\}$ form a vector frame (which is not necessarily orthonormal), the reciprocal frame is determined by [45]

$$e^i = (-1)^{i-1} e_1 \wedge e_2 \wedge \ldots \wedge \check{e}_i \wedge \ldots \wedge_n e^{-1} \tag{A40}$$

$$e \equiv e_1 \wedge e_2 \wedge \ldots \wedge e_n \tag{A41}$$

and the check on $\check{e}$ denotes that this term is missing from the expression. As usual, the two frames are related by

$$e_j \cdot e^k = \delta_j^k. \tag{A42}$$

From this frame, the multivector derivative $\partial_X$ in Equation (A39) is defined by

$$\partial_X \equiv \sum_{i<...<j} e^i \wedge ... \wedge e^j (e^j \wedge ... \wedge e_i) * \partial_X. \tag{A43}$$

The Leibnitz rule can be written in the form

$$\partial_X(AB) = \dot{\partial}_X \dot{A} B + \dot{\partial}_X A \dot{B}, \tag{A44}$$

where the overdot indicates the scope of the multivector derivative.

We have from (A39) and (A43)

$$\partial_X \langle XA \rangle = P_X(A), \ \ \partial_X \langle \tilde{X} A \rangle = P_X(\tilde{A}), \tag{A45}$$

where $P_X(A)$ is the projection of $A$ onto the grades contained in $X$. These results are combined using Leibnitz's rule to give

$$\partial_X \langle X\tilde{X} \rangle = \dot{\partial}_X \langle \dot{X}\tilde{X} \rangle + \dot{\partial}_X \langle X\dot{\tilde{X}} \rangle = 2\tilde{X}. \tag{A46}$$

For a vector argument $x$ and a constant vector $a$, (A39) and (A45) yield

$$a \cdot \partial_x x = a = \partial_x (x \cdot a). \tag{A47}$$

For a vector variable $a = a^\mu \gamma_\mu = a \cdot \gamma_\mu \gamma^\mu = a \cdot \gamma^\mu \gamma_\mu$, where $\gamma^\mu$ constitutes the reciprocal basis and satisfies $\gamma_\mu \cdot \gamma^\nu = \delta_\mu^\nu$, the vector derivative can be defined as

$$\partial_a \equiv \gamma^\mu \frac{\partial}{\partial a^\mu}, \tag{A48}$$

For the derivative with respect to a spacetime position vector $x$, we use the symbol $\nabla \equiv \partial_x = \gamma^\mu \frac{\partial}{\partial x^\mu}$, if $x = x^\mu \gamma_\mu$. From (A47) and (A48), we can obtain useful results

$$\partial_a = \partial_b b \cdot \partial_a = \gamma^\mu \gamma_\mu \cdot \partial_a. \tag{A49}$$

Some results for the derivative with respect to position vector $x$ in an n-dimensional space are [45]

$$\partial_x (x \cdot A_r) = r A_r, \tag{A50a}$$

$$\partial_x (x \wedge A_r) = (n-r) A_r, \tag{A50b}$$

$$\dot{\partial}_x A_r \dot{x} = (-1)^r (n-2r) A_r. \tag{A50c}$$

From (A21) and (A50a), an r-blade $A_r$ can be expressed as

$$A_r = \frac{1}{r!} \partial_{a_1} \wedge ... \wedge \partial_{a_r} (a_r \wedge ... \wedge a_1) * A_r. \tag{A51}$$

When considering a vector argument $x$ and a constant vector $a$, (A39) becomes the definition of the directional derivative $a \cdot \nabla$, thus

$$a \cdot \nabla F = a \cdot \partial_x F(x) = \frac{d}{d\tau} F(x + a\tau) \Big|_{\tau=0} = \lim_{\tau \to 0} \frac{F(x + a\tau) - F(x)}{\tau}. \tag{A52}$$

Then, the general vector derivative can be obtained from the directional derivative using (A49) as

$$\nabla F = \partial_x F(x) = \partial_a a \cdot \partial_x F(x). \tag{A53}$$

The directional derivative (A52) produces from $F$ a tensor field termed differential of $F$, denoted variously by

$$\underline{f}(a) = F_a \equiv a \cdot \nabla F. \tag{A54}$$

The underbar notation serves to indicate that $\underline{f}(a)$ is a linear function of $a$. This induced linear function is very important for us to describe the apparatus of GC for handling transformations of spacetime and the induced transformations of multivector fields on spacetime.

Suppose there is a diffeomorphism that transforms each point $x$ in some region of spacetime into another point $x'$ as

$$x' = f(x). \tag{A55}$$

This induces a linear transformation of tangent vectors at $x$ to tangent vectors at $x'$ given by the differential

$$a'(x') = \underline{f}(a) = a \cdot \nabla f. \tag{A56}$$

If we regard $x$ as a map representing the ordering of points in spacetime, then $x'$ can be interpreted as a different map, or a remapping of the same spacetime. The transformation $f$ also induces an adjoint transformation $\bar{f}$, which takes a tangent vector $b'$ at $x'$ back to a tangent vector $b$ at $x$, as defined by

$$b(x) = \bar{f}(b') \equiv \partial_x f(x) \cdot b'(x'). \tag{A57}$$

The differential and its adjoint are related by

$$b' \cdot \underline{f}(a) = a \cdot \bar{f}(b'). \tag{A58}$$

By using this relation, we can find one transformation from another by

$$\bar{f}(a) = \partial_b b \cdot \bar{f}(a) = \partial_b(\underline{f}(b) \cdot a), \tag{A59}$$
$$\underline{f}(a) = \partial_b b \cdot \underline{f}(a) = \partial_b(\bar{f}(b) \cdot a). \tag{A60}$$

In addition to the induced linear transformations $\underline{f}(a)$ and $\bar{f}(a)$ of tangent vectors, by the rule of direct substitution, (A55) can also induce a transformation of a multivector field $F(x)$ defined by

$$F'(x') \equiv F'(f(x)) = F(x), \tag{A61}$$

in which directional derivatives of the two functions are related by the chain rule

$$a \cdot \nabla F = a \cdot \partial_x F'(f(x)) \tag{A62}$$
$$= (a \cdot \nabla_x f(x)) \cdot \partial_{x'} F'(x') \tag{A63}$$
$$= \underline{f}(a) \cdot \nabla' F' = a \cdot \bar{f}(\nabla') F' \tag{A64}$$
$$= a' \cdot \nabla' F'. \tag{A65}$$

The operator identity is

$$a \cdot \nabla = \underline{f}(a) \cdot \nabla' = a \cdot \bar{f}(\nabla') = a' \cdot \nabla'. \tag{A66}$$

Differentiation with respect to the vector $a$ yields

$$\nabla = \bar{f}(\nabla') \ \text{ or } \ \nabla' = \bar{f}^{-1}(\nabla). \tag{A67}$$

Now is an opportune moment to discuss linear algebra. In fact, GC enables us to carry out coordinate-free calculations in linear algebra, eliminating the need for matrices. Every linear transformation $\underline{f}$ on spacetime has a unique extension to a linear function on the whole STA, called outermorphism. For arbitrary multivectors $A$, $B$, and any scalar $\alpha$, the outermorphism is defined by the property

$$\underline{f}(A \wedge B) = \underline{f}(A) \wedge \underline{f}(B), \ \ \underline{f}(\alpha) = \alpha. \tag{A68}$$

It follows that for an $r$-blade $A_r = a_1 \wedge ... \wedge a_r$,

$$\underline{f}(A_r) = \underline{f}(a_1) \wedge ... \wedge \underline{f}(a_r). \tag{A69}$$

Since the outermorphism preserves the outer product, it also preserves grade:

$$\underline{f}(\langle A \rangle_r) = \langle \underline{f}(A) \rangle_r \tag{A70}$$

for any multivector $A$. This implies that $\underline{f}$ alters the pseudoscalar $i$ only by a scalar multiple:

$$\underline{f}(i) = (\det \underline{f}) i, \tag{A71}$$

which defines the determinant of $\underline{f}$. The product of two linear transformations $\underline{h} = \underline{g}\underline{f}$ also applies to their outermorphisms. It follows from (A71) that

$$\begin{aligned} \underline{h}(i) = \underline{g}(\underline{f}(i)) &= (\det \underline{f})\underline{g}(i) = (\det \underline{f})(\det \underline{g}) i && \text{(A72)} \\ \det(\underline{g}\underline{f}) &= \det \underline{g} \det \underline{f} && \text{(A73)} \\ \det \underline{f}^{-1} &= (\det \underline{f})^{-1}. && \text{(A74)} \end{aligned}$$

Every $\underline{f}$ has an adjoint $\bar{f}$, which can be extended to an outmorphism denoted by the same symbol

$$\langle A\bar{f}(B) \rangle = \langle B\underline{f}(A) \rangle \tag{A75}$$

for any multivectors $A$ and $B$. Unlike the outer product, the inner product is not generally preserved by outermorphisms. However, it obeys the law

$$\begin{aligned} A_r \cdot \bar{f}(B_s) &= \bar{f}\Big(\underline{f}(A_r) \cdot B_s\Big) \quad \text{for } r \le s, && \text{(A76)} \\ \underline{f}(A_r) \cdot B_s &= \underline{f}(A_r \cdot \bar{f}(B_s)) \quad \text{for } r \ge s. && \text{(A77)} \end{aligned}$$

From these identities, we can construct a formula for the inverse of a linear transformation. Consider a multivector $B$, lying entirely in the algebra defined by the pseudoscalar $i$, we have

$$\underline{f}(i)B = (\det \underline{f}) iB = \underline{f}(i\bar{f}(B)). \tag{A78}$$

Replacing $iB$ by $A$, we find

$$(\det \underline{f})A = \underline{f}(i\bar{f}(i^{-1}A)) \tag{A79}$$

with a similar result holding for the adjoint. It follows immediately that

$$\begin{aligned} \underline{f}^{-1}(A) &= i\bar{f}(i^{-1}A)(\det \underline{f})^{-1}, && \text{(A80)} \\ \bar{f}^{-1}(A) &= i\underline{f}(i^{-1}A)(\det \underline{f})^{-1}. && \text{(A81)} \end{aligned}$$

Once again, one great advantage of GC is that it eliminates unnecessary conceptual barriers between classical, quantum, and relativistic physics.

**Appendix B. Stress–Energy Tensor Derived from Dirac Theory**

In this Appendix, we show that the stress–energy tensor derived from the Dirac equation for free spin-$\frac{1}{2}$ particles can be decomposed into the sum of a symmetric part and an antisymmetric part. The symmetric part represents a classical pressureless ideal fluid, while the antisymmetric part represents the contribution of quantum potential energy. As a result, when spin is zero, the stress–energy tensor becomes that of a pressureless ideal fluid. We follow Hestenes's work [18], except that we set $\beta = 0$ for free spin-$\frac{1}{2}$ particles. As mentioned in our previous paper, the spinor field at spacetime point $x$ takes the form [14]

$$\psi(x) = \rho(x)^{1/2} R(x), \tag{A82}$$

where $\rho(x)$ is a scalar, representing the proper probability density and $R(x)$ is a rotor (Lorentz rotation) satisfying $R\tilde{R} = 1$. The rotor $R$ can be used to transform a fixed frame $\{\gamma_\mu\}$ into a new frame $\{e_\mu\}$

$$e_\mu = R\gamma_\mu \tilde{R}. \tag{A83}$$

We identify $v = e_0$ as the proper velocity associated with the expected history $x(\tau)$ of a particle, $v = \frac{dx}{d\tau}$. Our present objective is to derive a stress–energy tensor for a pressureless ideal fluid from the Dirac theory. The desired form of the tensor is as follows:

$$T_{\mu\nu} = \rho_m v_\mu v_\nu, \tag{A84}$$

where $v = v_\mu \gamma^\mu$ and $\rho_m = m\rho$, as explained in the previous paper.

For later use, we define

$$\text{spin bivector} : S \;=\; \frac{1}{2}R\gamma_2\gamma_1\tilde{R} = \frac{1}{2}e_2e_1 \tag{A85}$$

$$\text{spin density trivector} : S_3 \;=\; \frac{1}{2}\psi i\gamma_3\tilde{\psi} = \frac{1}{2}\rho e_2e_1 v = \rho S v. \tag{A86}$$

The Dirac equation is

$$\nabla\psi i\gamma_3 = m\psi. \tag{A87}$$

A stress–energy tensor $T(a)$ is a linear vector function of a vector variable $a$, which denotes a flux of energy–momentum through a hypersurface with normal $a$ at spacetime point $x$. The tensor $T(a)$ for the free Dirac field is given by [18,21]

$$T(a) = \gamma^\nu\langle a\partial_\nu\psi i\gamma_3\tilde{\psi}\rangle = \gamma^\nu a\cdot\langle\partial_\nu\psi i\gamma_3\tilde{\psi}\rangle_1. \tag{A88}$$

This tensor is not symmetric, and its adjoint (transposed tensor) is

$$\bar{T}(a) = \partial_b\langle T(b)a\rangle = \langle a\cdot\nabla\psi i\gamma_3\tilde{\psi}\rangle_1. \tag{A89}$$

We can decompose $T(a)$ into a sum of a symmetric part and an anti-symmetric part as [45]

$$T(a) = T_S(a) + T_A(a), \tag{A90}$$

where

$$T_S(a) = \frac{1}{2}(T(a) + \bar{T}(a)) = \frac{1}{2}\partial_a(a\cdot T(a)), \tag{A91}$$

$$T_A(a) = \frac{1}{2}(T(a) - \bar{T}(a)) = \frac{1}{2}a\cdot(\partial_b\wedge T(b)) = -\frac{1}{2}a\cdot(\partial_b\wedge\bar{T}(b)). \tag{A92}$$

From (A86) and (A89), we write

$$\begin{aligned}\partial_a\wedge\bar{T}(a) &= \dot{\nabla}\wedge\langle\dot{\psi}i\gamma_3\tilde{\psi}\rangle_1\\ &= \frac{1}{2}\langle\nabla\psi i\gamma_3\tilde{\psi} - \gamma^\mu(\psi i\gamma_3(\partial_\mu\psi)^\sim)\rangle_2\\ &= -\frac{1}{2}\langle\nabla\psi i\gamma_3\tilde{\psi} + \gamma^\mu(\psi i\gamma_3(\partial_\mu\psi)^\sim)\rangle_2\\ &= -\frac{1}{2}\langle\nabla(\psi i\gamma_3\tilde{\psi})\rangle_2\\ &= -\nabla\cdot S_3,\end{aligned} \tag{A93}$$

where we have used the fact that the Dirac Equation (A87) implies $\langle\nabla\psi i\gamma_3\tilde{\psi}\rangle_2 = 0$. Thus, according to (A92) and (A93), the anti-symmetric part of $T(a)$ can be written as

$$T_A(a) = \frac{1}{2}a\cdot(\nabla\cdot S_3) = \frac{1}{2}(a\wedge\nabla)\cdot S_3. \tag{A94}$$

We are now ready to define the local energy–momentum $p$, one of the most fundamental quantities of the Dirac theory, as [18]

$$\rho p = T(v) = v^\mu T_\mu, \tag{A95}$$

where $v = v^\mu \gamma_\mu$ and $T_\mu = T(\gamma_\mu)$ are understood. Now $T_\mu$ can be decomposed in the form [18]

$$T_\mu = \rho v_\mu p + N_\mu, \tag{A96}$$

where $N_\mu = N(\gamma_\mu)$ describes the flow of energy momentum normal to the velocity. This can be verified from (A95) and (A96), which reads $N(v) = N(v^\mu \gamma_\mu) = v^\mu N_\mu = 0$, since $v^\mu v_\mu = v^2 = 1$.

According to the conservation of energy and momentum, the divergence of a stress–energy tensor must be equal to the force density exerted on the particle. This relationship holds true even for free particles, where no external forces are acting on them. We thus write

$$\dot{T}(\dot{\nabla}) = \dot{\bar{T}}(\dot{\nabla}) = \dot{\bar{T}}(\gamma^\mu \dot{\partial}_\mu) = \partial_\mu \bar{T}^\mu = 0, \tag{A97}$$

where we have used the fact that the divergence of the anti-symmetric part of the stress–energy tensor vanishes. This is because, according to (A94), $\dot{T}_A(\dot{\nabla}) = 0$.

One advantage of STA is that the formulations of physical laws in Dirac theory, such as conservation laws and dynamics, are expressed in the same form as in classical mechanics. From (A83), it follows that

$$a \cdot \nabla e_\mu = \Omega(a) \times e_\mu, \tag{A98}$$

where

$$\Omega(a) = 2a \cdot \nabla R\tilde{R} \tag{A99}$$

is a bivector valued function of a vector variable, which can be explained as the angular velocity of the frame $\{e_\mu\}$ rotates in the $a$ direction, and $A \times B = \frac{1}{2}(AB - BA)$. From (A88) and (A96), we write

$$T_{\mu\nu} = T_\mu \cdot \gamma_\nu = \gamma_\mu \cdot \langle \partial_\nu \psi i \gamma_3 \tilde{\psi} \rangle_1 = \rho v_\mu p_\nu + N_{\mu\nu}, \tag{A100}$$

where $p_\mu = p \cdot \gamma_\mu$ and $N_{\mu\nu} = N_\mu \cdot \gamma_\nu$. To express $T_{\mu\nu}$ with observables, we need to calculate $\partial_\nu \psi i \gamma_3 \tilde{\psi}$ from (2). We have

$$\begin{aligned}
\partial_\nu \psi i \gamma_3 \tilde{\psi} &= \partial_\nu (\rho^{1/2} R) i \gamma_3 \rho^{1/2} \tilde{R} \\
&= \frac{1}{2} \partial_\nu \rho R i \gamma_3 \tilde{R} + \rho \partial_\nu R i \gamma_3 \tilde{R} \\
&= (\partial_\nu \ln \rho) S_3 + \Omega_\nu S_3 \\
&= (\partial_\nu \ln \rho) \rho S v + \rho \Omega_\nu S v \\
&= \rho (W_\nu - \partial_\nu S + \Omega_\nu S) v,
\end{aligned} \tag{A101}$$

Here, $\Omega_\nu = \Omega(\gamma_\nu) = 2\partial_\nu R\tilde{R}$; $W_\nu$ represents a bivector defined as

$$W_\nu \equiv \frac{1}{\rho} \partial_\nu (\rho S), \tag{A102}$$

which is called quantum potential, and this holds a pivotal position in quantum mechanics in the approach of causal interpretation [18]. The vector part of (A101) gives

$$\begin{aligned}
\langle \partial_\nu \psi i \gamma_3 \tilde{\psi} \rangle_1 &= \Omega_\nu \cdot S_3 \\
&= \rho (W_\nu \cdot v - \partial_\nu S \cdot v + \Omega_\nu \cdot S v + (\Omega_\nu \times S) \cdot v) \\
&= \rho (W_\nu \cdot v + \Omega_\nu \cdot S v),
\end{aligned} \tag{A103}$$

where we have used $\partial_\nu S = \Omega_\nu \times S$, which can be proved as follows

$$\begin{aligned}\partial_\mu S &= \frac{1}{2}\partial_\mu(R\gamma_2\gamma_1\tilde{R}) \\ &= \frac{1}{2}((\partial_\mu R)\gamma_2\gamma_1\tilde{R} + R\gamma_2\gamma_1\partial_\mu\tilde{R}) \\ &= \frac{1}{2}(\Omega_\mu S - S\Omega_\mu) \\ &= \Omega_\mu \times S.\end{aligned} \tag{A104}$$

We thus obtain from (A88), (A95) and (A103),

$$T(v) = v \cdot \langle \partial_\nu \psi i \gamma_3 \tilde{\psi} \rangle_1 \gamma^\nu = (v \wedge \Omega_\nu) \cdot S_3 \gamma^\nu = \rho \Omega_\nu \cdot S \gamma^\nu = \rho p, \tag{A105}$$

which means

$$p_\nu = \Omega_\nu \cdot S. \tag{A106}$$

Finally, from (A100), (A103) and (A106), we obtain [18]

$$T_{\mu\nu} = \rho v_\mu p_\nu + \rho(v \wedge \gamma_\mu) \cdot W_\nu. \tag{A107}$$

To achieve the desired result (A84), we need to identify the constraints that would align the momentum and velocity (i.e., $v \wedge p = 0$), and ensure the symmetry of the stress–energy tensor (i.e., $T(a) = \bar{T}(a)$ or $T_{\mu\nu} = T_{\nu\mu}$).

Let us first multiply the Dirac Equation (A87) on the right by $\tilde{\psi}$ to obtain

$$\nabla \psi \gamma_2 \gamma_1 \tilde{\psi} = m\rho v. \tag{A108}$$

Next, we evaluate $\nabla \psi \gamma_2 \gamma_1 \tilde{\psi}$ by substituting $\psi$ from (2), resulting in

$$\begin{aligned}\gamma^\mu(\partial_\mu \psi \gamma_2 \gamma_1 \tilde{\psi}) &= \gamma^\mu((\partial_\mu \sqrt{\rho}) R \gamma_2 \gamma_1 \tilde{\psi} + \sqrt{\rho}(\partial_\mu R)\gamma_2\gamma_1\tilde{\psi}) \\ &= \gamma^\mu((\partial_\mu \rho) S + \rho \Omega_\mu S) \\ &= (\nabla \rho) S + \rho \gamma^\mu \Omega_\mu S.\end{aligned} \tag{A109}$$

The vector part of this equation gives

$$\begin{aligned}\langle \nabla \psi \gamma_2 \gamma_1 \tilde{\psi} \rangle_1 &= \langle (\nabla \rho) S + \rho \gamma^\mu \Omega_\mu S \rangle_1 \\ &= (\nabla \rho) \cdot S + \rho \gamma^\mu \Omega_\mu \cdot S + \rho \gamma^\mu \cdot (\Omega_\mu \times S) \\ &= (\nabla \rho) \cdot S + \rho p + \rho \nabla \cdot S \\ &= \nabla \cdot (\rho S) + \rho p.\end{aligned} \tag{A110}$$

From (A108) and (A110), we obtain

$$p = mv - \frac{1}{\rho}\nabla \cdot (\rho S). \tag{A111}$$

Decomposing the stress–energy tensor into its symmetric and anti-symmetric parts in terms of observables proves to be highly advantageous. In the main text, we specifically utilize the adjoint form outlined in this Appendix, hence we express $\bar{T}_{\mu\nu}$ as derived from (A107) as follows:

$$\bar{T}_\mu = \rho p_\mu v + [\partial_\mu (S_3 v)] \cdot v. \tag{A112}$$

By substituting $p$ given by (A111), we have

$$\begin{aligned}\bar{T}(a) =& \rho a \cdot pv + [a \cdot \nabla(S_3 v)] \cdot v \\ =& \rho\left[ma \cdot v - \frac{1}{\rho}(a \wedge \nabla) \cdot (S_3 v)\right]v + [a \cdot \nabla(S_3 v)] \cdot v \\ =& \rho_m a \cdot vv + [a \cdot \nabla(S_3 v)] \cdot v - (a \wedge \nabla) \cdot (S_3 v)v,\end{aligned} \tag{A113}$$

where $\rho_m = \rho m$ represents the proper mass density. We see that if $S_3 = 0$, we promptly obtain the desired expression for a stress–energy tensor that is appropriate to describe a pressureless ideal fluid

$$T(a) = \bar{T}(a) = \rho_m a \cdot vv. \tag{A114}$$

**Appendix C. GTG and Matter**

By virtue of GC, GTG is constructed such that gravitational effects are described by a pair of gauge fields, $\bar{h}(a) = \bar{h}(a, x)$ and $\omega(a) = \omega(a, x)$, defined over a flat Minkowski background spacetime [26], where $x$ is the STA position vector, which is usually suppressed for short.

The first of them, $\bar{h}(a)$, is a position-dependent linear function mapping the vector argument $a$ to vectors. The introduction of $\bar{h}(a)$ ensures covariance of the equations under arbitrary local displacements (or an arbitrary remapping $x' = f(x)$) of the matter fields in the background spacetime. In order to understand the physical meaning of the $\bar{h}$ field, we first define the covariant displacement transformation as

$$M(x) \xrightarrow{x'=f(x)} M'(x) = M(x'), \tag{A115}$$

so that the equations $A(x) = B(x)$ and $A(x') = B(x')$ have exactly the same physical content. Suppose we have a vector field $b(x) = \nabla\phi(x)$, where $\phi(x)$ is a scalar field that is already covariant under displacement, i.e., $\phi'(x) = \phi(x')$. Now can we write $b'(x) = b(x')$ or $\nabla\phi'(x) = \nabla_{x'}\phi(x')$? Using the chain rule, we find

$$\begin{aligned}a \cdot \nabla\phi'(x) &= a \cdot \nabla\phi(x') \\ &= (a \cdot \nabla f(x)) \cdot \nabla_{x'}\phi(x') \\ &= \underline{f}(a) \cdot \nabla_{x'}\phi(x') \\ &= a \cdot \bar{f}(\nabla_{x'})\phi(x'),\end{aligned} \tag{A116}$$

where $\underline{f}(a) = a \cdot \nabla f(x)$ is a linear function of $a$ and an arbitrary function of $x$, and $\bar{f}(\nabla_{x'}) = \nabla f(x) \cdot \nabla_{x'}$, and we call $\bar{f}$ the adjoint of $\underline{f}$, satisfying $a \cdot \underline{f}(b) = \bar{f}(a) \cdot b$, or $\bar{f}(a) = \partial_b\langle \underline{f}(b)a\rangle$. It follows that $\nabla\phi'(x) = \bar{f}(\nabla_{x'}\phi(x'))$, or

$$\nabla_x = \bar{f}(\nabla_{x'}) \quad \text{and} \quad \bar{f}^{-1}(\nabla_x) = \nabla_{x'}, \tag{A117}$$

which shows us that $b(x)$ is not covariant under displacement. In order to make objects such as $b(x)$ covariant, we must introduce a position-gauge field $\bar{h}(a, x)$, which is a linear function of $a$ and arbitrary function of $x$, so that

$$\bar{h}(a, x) \xrightarrow{x'=f(x)} \bar{h}'(a, x) = \bar{h}(\bar{f}^{-1}(a), x'). \tag{A118}$$

Now, if we redefine $b(x) = \bar{h}(\nabla\phi(x))$, then

$$\begin{aligned}b(x) = \bar{h}(\nabla\phi(x)) \xrightarrow{x'=f(x)} b'(x) &= \bar{h}'(\nabla\phi'(x)) \\ &= \bar{h}(\bar{f}^{-1}(\nabla\phi'(x))) \\ &= \bar{h}(\nabla_{x'}\phi(x')) = b(x'),\end{aligned} \tag{A119}$$

which becomes covariant. The $\bar{h}(a)$ field plays the same role as vierbein in the tensor calculus approach of gauge theory of gravity [25,26]. For later use, we give the relationship

between $\bar{h}(a)$ and the metric tensor $g_{\mu\nu}$ in GR. We define a position gauge invariant directional derivative as [26,28]

$$L_a = a \cdot \bar{h}(\nabla) = \underline{h}(a) \cdot \nabla, \tag{A120}$$

where $\underline{h}$ is the adjoint of $\bar{h}$ defined by $\underline{h}(b) \equiv \partial_c(\bar{h}(c) \cdot b)$, and $a$ is an invariant vector (i.e., for $x' = f(x)$, $a(x)$ is transformed to $a'(x) = a(x')$). So, $\underline{h}$ maps tangent vectors to tangent vectors and $\bar{h}$ maps cotangent vectors to cotangent vectors. For a given set of coordinates $\{x^\mu, \mu = 0,1,2,3\}$, we introduce the basis vectors

$$e_\mu \equiv \frac{\partial x}{\partial x^\mu}, \quad e^\mu \equiv \nabla x^\mu, \tag{A121}$$

which satisfy $e_\mu \cdot e^\nu = \delta^\nu_\mu$. From these vectors, we further define vectors

$$g_\mu \equiv \underline{h}^{-1}(e_\mu), \quad g^\mu \equiv \bar{h}(e^\mu). \tag{A122}$$

These vectors satisfy the relation

$$g_\mu \cdot g^\nu = \underline{h}^{-1}(e_\mu) \cdot \bar{h}(e^\nu) = e_\mu \cdot \bar{h}^{-1}(\bar{h}(e^\nu)) = e_\mu \cdot e^\nu = \delta^\nu_\mu. \tag{A123}$$

The metric tensor is then given by

$$g_{\mu\nu} \equiv g_\mu \cdot g_\nu. \tag{A124}$$

Let $x(\tau)$ be a time-like curve (where $\tau$ is the proper time), a mapping $f : x \to x' = f(x)$ induces the transformation

$$\dot{x} = \frac{dx}{d\tau} \to \dot{x}' = \frac{dx'}{d\tau} = \frac{dx}{d\tau} \cdot \nabla f(x). \tag{A125}$$

Comparing Equation (A125) with Equation (A120), we introduce an invariant velocity $v = v(x(\tau))$ as [28]

$$\dot{x} = \underline{h}(v), \quad v = \underline{h}^{-1}(\dot{x}). \tag{A126}$$

From the known formula $dx = dx^\mu e_\mu$, the invariant normalization $v^2 = 1$ induces the invariant line element on a time-like curve in GR

$$d\tau^2 = [\underline{h}^{-1}(dx)]^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{A127}$$

Another gauge field, $\omega(a)$, is a position-dependent linear function mapping the vector argument $a$ to bivectors. Its introduction ensures covariance of the equations of physics under local Lorentz rotations described by the rotor $R$. Under local Lorentz rotations, the multivector $M$ transforms as $M' = RM\tilde{R}$ and the spinor $\psi$ transforms as $\psi' = R\psi$. To ensure covariance of the quantities like $\bar{h}(\nabla)M$ and $\bar{h}(\nabla)\psi$ under local Lorentz rotations, $\bar{h}(\nabla) = \bar{h}(\partial_a)a \cdot \nabla$ should be replaced by a covariant derivative $D$ [26]. To achieve this, we focus attention on $a \cdot \nabla\psi' = a \cdot \nabla(R\psi)$ and write

$$a \cdot \nabla(R\psi) = Ra \cdot \nabla\psi + (a \cdot \nabla R)\psi. \tag{A128}$$

Clearly, the presence of the term $(a \cdot \nabla R)\psi$ renders the operator $a \cdot \nabla$ non-covariant. Since the rotor $R$ satisfies $R\tilde{R} = 1$, we find that

$$a \cdot \nabla R\tilde{R} + Ra \cdot \nabla\tilde{R} = 0, \tag{A129}$$

which implies

$$a \cdot \nabla R\tilde{R} = -Ra \cdot \nabla\tilde{R} = -(a \cdot \nabla R\tilde{R})^\sim. \tag{A130}$$

Hence, $a \cdot \nabla R\tilde{R}$ is equal to minus its reverse and thus must be a bivector in the STA. We can therefore rewrite (A128) as

$$a \cdot \nabla(R\psi) = Ra \cdot \nabla\psi + \frac{1}{2}(2a \cdot \nabla R\tilde{R})(R\psi). \tag{A131}$$

This suggests that to achieve a covariant derivative we must add a connection term to $a \cdot \nabla$ to construct an operator

$$D_a\psi = a \cdot \nabla\psi + \frac{1}{2}\Omega(a)\psi \tag{A132}$$

which must be covariant under local Lorentz rotations. The connection $\Omega(a) = \Omega(a, x)$ is a bivector valued linear function of $a$ with an arbitrary $x$ dependence. Under local rotations we expect that the operator $D_a$ will be unchanged in form, namely,

$$D'_a = a \cdot \nabla + \frac{1}{2}\Omega'(a), \tag{A133}$$

Here, we have used the fact that $a \cdot \nabla$ cannot change under local rotations. Nevertheless, the property that the covariant derivative must satisfy is

$$\begin{aligned} D'_a\psi' &= (a \cdot \nabla + \tfrac{1}{2}\Omega'(a))(R\psi) \\ &= RD_a\psi = R(a \cdot \nabla\psi + \tfrac{1}{2}\Omega(a)\psi). \end{aligned} \tag{A134}$$

From these identities, it follows that $\Omega(a)$ transforms as

$$\Omega'(a) = R\Omega(a)\tilde{R} - 2a \cdot \nabla R\tilde{R}. \tag{A135}$$

Now, we reassemble the covariant derivative (A132) with the term $\bar{h}(\partial_a)$ to form

$$D \equiv \bar{h}(\partial_a)D_a, \tag{A136}$$

and write

$$D\psi = \bar{h}(\partial_a)(a \cdot \nabla\psi + \frac{1}{2}\Omega(a)\psi). \tag{A137}$$

Note that the vector derivative $D$ is fully covariant, but $D_a$ is not, since it contains the $\Omega(a)$ field, which must transform in the same way as $a \cdot \nabla R\tilde{R}$ under displacement and thus picks up a term in $\underline{f}$ (A66). Recall the definition of the position gauge invariant directional derivative (A120), we thus define

$$a \cdot D\psi = a \cdot \bar{h}(\nabla)\psi + \frac{1}{2}\omega(a)\psi, \tag{A138}$$

where $\omega(a)$ is defined by

$$\omega(a) = \Omega(h(a)). \tag{A139}$$

It should be pointed out that the vector $a$ is declared to be position gauge-invariant, as stated below Equation (A120). Therefore, $D_a$ is related to $a \cdot D$ by

$$\begin{aligned} D_a\psi &= \underline{h}^{-1}(a) \cdot D\psi \\ &= \underline{h}^{-1}(a) \cdot \bar{h}(\nabla)\psi + \tfrac{1}{2}\omega(\underline{h}^{-1}(a))\psi \\ &= a \cdot \nabla\psi + \tfrac{1}{2}\Omega(a)\psi, \end{aligned} \tag{A140}$$

or simply

$$D_a = \underline{h}^{-1}(a) \cdot D. \tag{A141}$$

We thus also have

$$D\psi \equiv \partial_a\left(a \cdot \bar{h}(\nabla)\psi + \frac{1}{2}\omega(a)\psi\right). \tag{A142}$$

We are now ready to establish the form of the covariant derivatives of the observables formed from a spinor field. In general, such observables have the form

$$M = \psi\Gamma\tilde{\psi}, \tag{A143}$$

where $\Gamma$ is a constant multivector formed from combinations of $\gamma_0$, $\gamma_3$ and $i\sigma_3$. Under displacements $x' = f(x)$, the spinor fields are covariant: $\psi'(x) = \psi(x')$. Thus, the observable $M$ inherits its transformation properties from the spinor $\psi$,

$$M(x) \xrightarrow{x'=f(x)} M'(x) = M(x'), \tag{A144}$$

exactly the same form in (A115). Under rotations, the spinor transforms as $\psi' = R\psi$. So it follows from (A143) that, under rotations, $M$ transforms as

$$M(x) \xrightarrow{R} M' = \psi'\Gamma\tilde{\psi}' = R\psi\Gamma\tilde{\psi}\tilde{R} = RM\tilde{R}. \tag{A145}$$

To achieve the fully covariant derivative for $M$, we first write

$$L_a M = L_a\psi\Gamma\tilde{\psi} + \psi\Gamma(L_a\psi)^{\sim}. \tag{A146}$$

This equation is a displacement covariant. We simply replace $L_a$ with $a \cdot D$ to obtain

$$\begin{aligned} a \cdot DM &= a \cdot D\psi\Gamma\tilde{\psi} + \psi\Gamma(a \cdot D\psi)^{\sim} \\ &= L_a(\psi\Gamma\tilde{\psi}) + \omega(a) \times (\psi\Gamma\tilde{\psi}) \\ &= L_a M + \omega(a) \times M. \end{aligned} \tag{A147}$$

Note the difference between the forms of $a \cdot D$ acting on $M$ and $\psi$.

The field strength corresponding to the $\omega(a)$ gauge field is defined by

$$[a \cdot D, b \cdot D]\psi = \frac{1}{2}\mathcal{R}(a \wedge b)\psi, \tag{A148}$$

where

$$\mathcal{R}(a \wedge b) \equiv L_a\omega(b) - L_b\omega(a) + \omega(a) \times \omega(b), \tag{A149}$$

$a$ and $b$ are constant vectors. The Ricci tensor $\mathcal{R}(a)$, Ricci scalar $\mathcal{R}$, and Einstein tensor $\mathcal{G}(a)$ are defined, respectively, as

$$\mathcal{R}(a) = \partial_b \cdot \mathcal{R}(b \wedge a), \tag{A150}$$

$$\mathcal{R} = \partial_a \cdot \mathcal{R}(a), \tag{A151}$$

$$\mathcal{G}(a) = \mathcal{R}(a) - \frac{1}{2}a\mathcal{R}. \tag{A152}$$

The overall action integral is of the form

$$I = \int |d^4x| \det(\mathsf{h})^{-1}(\frac{1}{2}\mathcal{R} - \kappa\mathcal{L}_m), \tag{A153}$$

where $\mathcal{L}_m$ describes the matter content and $\kappa = 8\pi$. In this paper, we adopt the covariant $\mathcal{L}_m = Di\gamma_3\tilde{\psi} - m\psi\tilde{\psi}$ from Dirac theory to describe the macroscopic matter, which has been well-studied [26,27,30] for electrons. From (A153), we obtain the following equations that describe the field coupled self-consistently to gravity [27]:

$$\text{torsion:} \quad D \wedge \bar{h}(a) = \kappa\bar{h}(a) \cdot S, \tag{A154}$$

$$\text{Einstein:} \quad \mathcal{G}(a) = \kappa\mathcal{T}(a) \tag{A155}$$

$$\text{Dirac:} \quad D\psi i\sigma_3 = m\psi\gamma_0, \tag{A156}$$

where $D \wedge \bar{h}(a)$ is the gravitational torsion which is determined by the trivector spin density $S \equiv \frac{1}{2}\psi i\gamma\tilde{\psi}$, $\kappa = 8\pi$, and

$$\mathcal{T}(a) = \langle a \cdot D\psi i\gamma_3\tilde{\psi}\rangle_1 \tag{A157}$$

is the matter stress–energy tensor. We can solve Equation (A154) for $\omega(a)$ to obtain [27]

$$\omega(a) = \omega'(a) + \frac{1}{2}\kappa a \cdot S = -H(a) + \frac{1}{2}a \cdot [\partial_b \wedge H(b)] + \frac{1}{2}\kappa a \cdot S, \tag{A158}$$

this defines $\omega'(a)$ as the $\omega$-function in the absence of torsion, and

$$H(a) \equiv \bar{h}(\dot{\nabla} \wedge \dot{\bar{h}}^{-1}(a)) = -\bar{h}(\dot{\nabla}) \wedge \dot{\bar{h}}(\bar{h}^{-1}(a)), \tag{A159}$$

where 'overdot' notation is employed to denote the scope of a differential operator. This proves convenient if we employ the primed symbols to denote the torsion-free part of the curvature tensors, and we obtain [27]

$$\begin{aligned} \mathcal{R}(a \wedge b) &= \mathcal{R}'(a \wedge b) + \frac{1}{4}\kappa^2[(a \wedge b) \cdot S] \cdot S \\ &\quad -\frac{1}{2}\kappa[(a \wedge b) \cdot D] \cdot S, \qquad \text{(A160)} \\ \mathcal{R}(a) &= \mathcal{R}'(a) + \frac{1}{2}\kappa^2(a \cdot S) \cdot S \\ &\quad -\frac{1}{2}\kappa a \cdot (D \cdot S), \qquad \text{(A161)} \\ \mathcal{R} &= \mathcal{R}' + \frac{3}{2}\kappa^2 S^2. \qquad \text{(A162)} \end{aligned}$$

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